Posted Price and a Knockout Auction: Revenue Maximization Under the Threat of Collusion

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Abstract

I explore how a revenue-maximizing principal allocating a single good should optimally design his auction in the presence of collusion. The principal evaluates mechanisms according to the worst-case revenue that could arise from collusive or non-collusive play. The principal's optimal mechanism in the face of collusion is to post a price and run an efficient knockout auction in-house. This remains the optimal mechanism when the principal additionally hypothesizes that colluders are maximizing their joint surplus; with surplus-maximizing colluders, posting a price without running the knockout in-house is also an optimal mechanism.

1 Introduction

Many instances of auctions are part of long-run interactions between the bidders. Dynamic incentives may sustain complicated arrangements between the bidders, so bidders may not necessarily play auctions in the non-cooperative manner captured by a Bayes-Nash equilibrium. Instead, they may choose to collude. Collusion threatens the revenue promised by optimal auctions. Consider, for example, first-price or second-price auctions with reserve prices, which are optimal mechanisms when bidders' valuations are drawn independently from the same regular distribution. In contrast to a posted price, these auction formats incentivize bidders with high valuations to bid high to prevent others from getting the good and secure it for themselves instead. Collusion allows bidders to coordinate on entering low bids and suppresses competition for the good.

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To what extent collusion can undermine the principal's revenue was first explored in McAfee and McMillan (1992) and Graham and Marshall (1987). Both study how surplus-maximizing colluders should organize themselves to play particular auction formats. McAfee and McMillan (MM) consider collusion against a first-price auction with a reserve price, while Graham and Marshall (GM) study the second-price auction with a reserve price. In both papers, surplus-maximizing colluders allocate the good efficiently amongst themselves through a pre-auction knockout, which determines who gets the right to buy the good at the reserve price. Both MM and GM restrict the principal to setting a first-price auction or a second-price auction, respectively, so the principal can only respond to collusion by adjusting the reserve price of his auction format. Since colluders pay exactly the reserve price whenever their highest realized valuation exceeds it, the principal optimally chooses a reserve price equal to the optimal posted price given a single bidder whose valuation matches the colluders' highest realized valuation.

In this paper, the principal can freely choose the auction format rather than being constrained to adjusting the reserve price in a first-price or second-price auction. Additionally, the principal operates under weaker informational assumptions about what colluders can achieve. For example, the principal is sure that if collusion occurs, it is preferred to non-collusive play, and potential colluders must contend with the same informational frictions that affect non-collusive play, but beyond these constraints, the principal does not take a stand on the arrangements that colluders can form. In contrast to MM and GM, surplus-maximizing collusive arrangements here are not assumed to be feasible. As the principal does not fully understand the colluders' capabilities, he evaluates mechanisms based on their worst-case revenue, considering both non-collusive and collusive scenarios.

While the principal has the flexibility to design a complex mechanism, Theorem 1 establishes that the optimal mechanism is simple. The principal sets the posted price that would be optimal if he were facing a single bidder with a valuation equal to the maximum realized valuation of the colluders, and he runs a pre-auction knockout in-house to determine who gets the right to buy the good. In other words, when the principal can contemplate more complicated auction formats, the equilibrium outcome identified by GM and MM—assuming surplus-maximizing colluders and a constrained principal—remains globally optimal for the principal to induce.

The principal's ultimate goal is to maximize revenue, but it turns out that maximizing bidder surplus is an important instrumental goal. The principal's mechanism influences collusion in two crucial ways: (1) it determines what arrangements colluders *can* execute, and (2) it influences what arrangements colluders *want* to execute. At a high level, (1) is governed by the minimum prices bidders must collectively pay to acquire the good at different probabilities, while (2) is determined by the joint surplus generated through non-collusive play. Intuitively, if the non-collusive joint surplus is low, bidders find more collusive schemes to be attractive, alternative ways to play the mechanism, and they are more likely to collude. As long as the influence of the principal's mechanism through (1) remains largely unchanged, maximizing the surplus from non-collusive play is a valuable secondary objective. This secondary motive drives the principal to allocate the good efficiently whenever he sells it and fixes some features of the optimal mechanism's transfer rule.

After identifying the mechanism that maximizes non-collusive surplus while holding fixed (1), the principal optimizes over various minimum price schedules, considering only mechanisms that solve the surplus-maximization subproblem. The principal's outer problem over price schedules is equivalent to a single-bidder screening problem with a standard revenue-maximizing principal. Given Myerson (1981), the posted price emerges as part of the optimal mechanism.

When colluders can implement arrangements that maximize their joint surplus, the mechanism stated in Theorem 1 remains optimal, but now, the principal can also achieve the optimum by simply posting a price (Theorem 2). Since bidders are assumed to collude efficiently, the principal is indifferent between doing the knockout auction in-house and allowing colluders to carry it out on their own.

The principal's minimal model of collusion and worst-case evaluation of mechanisms can be interpreted as arising from uncertainty about the unmodeled, possibly unknown frictions colluders face. For example, it is well-documented that real collusive arrangements are sustained by attempts at monitoring. Colluders report their actions to establish their compliance with the collusive agreement. The frequency of these attempts varies across cartels, ranging from a couple of times a year to weekly as discussed in Harrington Jr (2006). For whatever reason (unknown to the principal and so outside the model), a cartel may be unable to sustain very frequent monitoring, and this may prevent them from implementing certain collusive arrangements. The principal may not have a full understanding of bidders' abilities to monitor each other and so may not know exactly what arrangements are feasible for colluders.

Similarly, colluding often entails some risk of detection and possible punishment by a regulator. Collusion, if it happens, must be profitable enough to outweigh this expected cost. The principal may not be able to accurately estimate this cost since detection, successful conviction, and the extent of fines ultimately depend on the details of how colluders interact and how much evidence they generate of their coordination. The principal may be unwilling to consider a detailed model of collusion due to limited knowledge about the exact technologies that enable coordination. Given these uncertainties about the other frictions that impede collusion, the principal chooses to focus on the revenue he can guarantee and evaluates mechanisms according to their worst-case revenue under non-collusive or collusive play.

This paper demonstrates how the principal can optimally respond to collusive threats by deliberately leaving surplus on the table. It is shown that the best way to achieve this is to ensure efficient allocation and commit to a single posted price.

1.1 Related Literature

In addition to McAfee and McMillan (1992) and Graham and Marshall (198), there is an existing literature on optimal auction design in the presence of collusion. This strand of the literature was initiated by Laffont and Martimort (1997) and Laffont and Martimort (2000). In these models, collusion is required to satisfy an interim revealed preference constraint, i.e., a necessary condition for collusion is that bidders prefer their interim payoff while colluding to their interim payoff while playing non-collusively. Because collusive arrangements form at the interim stage, the principal can focus without loss of optimality on "collusion-proof" mechanisms, mechanisms which are interim Pareto efficient and so leave no scope for collusion since any collusive arrangement will necessarily make some type of some bidder worse off.

Following Laffont and Martimort, Che and Kim (2006) study the issue of collusion-proof auction design in a very general setting and conclude that any payoff from a BNE can be implemented in a collusion-proof way. In short, the principal can totally insulate himself from collusion by manipulating the ex post transfers to deliver a constant payoff at every action profile. This powerful positive conclusion was softened by later work Che and Kim (2009) and Pavlov (2008), which note that in Che and Kim's construction of a collusion-proof mechanism, there was no opt-out action. Che and Kim (2009) and Pavlov (2008) study collusion-proof auction design but with the requirement of an opt-out action. I also adopt this requirement that the principal include an opt-out action for each player. Che and Kim (2009) and Pavlov (2008) show that under certain assumptions, the principal can still achieve his optimal payoff assuming no collusion, even in the presence of collusion.

In contrast to these papers, I consider a setting where bidders are playing many instances of the same auction over time, and the collusive arrangement is not formed anew with every instance of the auction. Collusion occurs if the bidders' collusive payoffs averaged over time exceed their non-collusive payoffs averaged over time, so bidders' preferences for colluding over not colluding are evaluated at the ex ante stage. I do not include a regulator to which the principal can report possible collusion, and the principal cannot vary his auction format in response to observations of bidder play. The current exercise is informative as a measure of what can be done when the principal simply sets an auction format, or as a strategy when the detection of collusion and dynamically changing the auction format may be prohibitively costly.

Ultimately, this paper provides a main result that is less positive than those of Che and Kim (2006), Che and Kim (2009), and Pavlov (2008). In my setting, the principal is unable to achieve his optimal payoff assuming no collusion. To cope with the presence of collusion, he is driven to compromise the allocation he would choose assuming no collusion. While the principal would not necessarily allocate the good efficiently in the optimal auction studied in Myerson (1981), facing collusion, the principal's optimal mechanism efficiently allocates the good whenever it is sold.

2 Model

N bidders participate in a single good auction designed by the principal. Each bidder has a valuation for the good v_i, which is drawn from an atomless distribution $F_i[0, 1]$ with full support and density f_i. Valuations are drawn independently and are each bidder's private information. I use V to refer to the space of all profiles of valuations, $[0, 1]^N$.

The principal designs an auction by specifying a space of actions for each bidder A_i, a mapping from action profiles to probabilities of allocating the good to each bidder $q : A \to [0, 1]^N$ where $\sum_i q_i(a) \leq 1$ for each a, and mappings from action profiles to transfers paid by each bidder $t_i : A \to \mathbb{R}$. Participation in the principal's auction is voluntary, so the auction must contain an opt-out action for each bidder that guarantees the bidder gets the good with 0 probability and pays nothing. Bidders have standard quasilinear utility, so playing action profile a results in payoff $q_i(a)v_i - t_i(a)$ for bidder i with valuation v_i.

The principal entertains two possible models for how bidders behave. On one hand, bidders may be playing *non-collusively*. This is modeled as playing the principal's preferred Bayes-Nash equilibrium of the auction. On the other hand, bidders may be playing *collusively*, which encompasses a rich set of behaviors. While colluding, bidders may coordinate their actions, reallocate the good between themselves, or exchange side payments. I model collusion as a (direct) side mechanism that bidders design to play the principal's mechanism.

The side mechanism consists of:

- a bid coordination scheme $\rho : V \to \Delta A$, which specifies how the collusive arrangement will play the principal's original mechanism given a profile of bidders' valuations,
- an allocation rule $\tilde{q} : V \to [0, 1]^N$ with $\sum_i \tilde{q}_i(v) \leq 1$, which associates each profile of valuations with a probability bidder i ultimately gets the good,

- a transfer rule $\tilde{t} : V \to \mathbb{R}^N$, which associates a profile of bidders' valuations with the transfer a particular bidder i must make to the collusive arrangement in excess of her payments to the principal.

The principal is agnostic about the exact nature of this side mechanism, but he knows that a few key frictions constrain it:

1. ***Feasibility.*** The side mechanism can only allocate the good with at most the probability that colluders acquire the good:

$$\sum_i \tilde{q}_i(v) \leq \sum_{\hat{a} \in A} \sum_i \rho(\hat{a}|v) q_i(\hat{a}) \quad \forall v \tag{Feas}$$

2. ***Private information between bidders.*** The same informational frictions exist when bidders collude as when they play non-collusively, so the side mechanism must incentivize truthful revelation of each bidder's private information:

$$\begin{aligned} &\mathbb{E}_{v_{-i}}[\tilde{q}_i(v)v_i - \tilde{t}_i(v) - \sum_{\hat{a}} \rho(\hat{a}|v) t_i(\hat{a})] \\ &\quad \geq \mathbb{E}_{v_{-i}}[\tilde{q}_i(\tilde{v}_i, v_{-i})v_i - \tilde{t}_i(\tilde{v}_i, v_{-i}) - \sum_{\hat{a}} \rho(\hat{a}|\tilde{v}_i, v_{-i}) t_i(\hat{a})] \quad \forall v_i, \tilde{v}_i, i \end{aligned} \tag{C-IC}$$

3. ***Revealed preference.*** Bidders only collude if they benefit from colluding over playing the mechanism non-collusively:

$$\mathbb{E}_{v_{-i}}[\tilde{q}_i(v)v_i - \tilde{t}_i(v) - \sum_{\hat{a}} \rho(\hat{a}|v) t_i(\hat{a})] \geq U_i^n \tag{EAIR}$$

where U_i^n is the expected payoff of the bidder i under the non-collusive equilibrium chosen by the principal.

4. ***No long-term deficit.*** Collusion is not subsidized by outside sources, so expected payments between bidders must be equal to 0.

$$\mathbb{E}_v[\sum_i \tilde{t}_i(v)] = 0 \tag{EABB}$$

The principal evaluates the auction according to its worst-case revenue across all conjectured models of play. Note that bidders playing the principal's non-collusive BNE is always feasible as a "collusive" scheme. Formally, the principal calculates for each mechanism

(A, q, t):

$$R_c(A, q, t) := \inf_{\rho, \tilde{q}, \tilde{t}} \quad \mathbb{E}_v[\sum_{\hat{a} \in A} \rho(\hat{a}|v) \sum_i t_i(a)]$$
$$\text{s.t. Feas, C-IC, EAIR, EABB}$$

Because the principal evaluates his mechanisms according to the worst-case revenue, a standard revelation principle argument goes through. By discarding actions that are not played in his preferred BNE (and are not the opt-out actions), the principal weakly increases the worst-case payoff since the set of action profiles available to colluders shrinks. As a result, it is without loss for the principal's mechanism to be a direct mechanism with additional actions $\{a_i^{optout}\}_i$ such that $q_i(a_i^{optout}, a_{-i}) = 0$ and $t_i(a_i^{optout}, a_{-i}) = 0$. From here, I take $A_i = V_i \cup a_i^{optout}$ to be the set of actions available to bidder i in the principal's mechanism. The principal's preferred BNE corresponds to truth-telling.

The principal's problem is

$$\begin{aligned}
\max_{q: A \to [0,1]^N, t: A \to \mathbb{R}^N} \quad & R_c(q, t) \\
\text{s.t.} \quad & \sum_i q_i(v) \leq 1 \quad \forall v \\
& \mathbb{E}_{v_{-i}}[q_i(v)v_i - t_i(v)] \geq \mathbb{E}_{v_{-i}}[q_i(\tilde{v}_i, v_{-i})v_i - t_i(\tilde{v}_i, v_{-i})] \quad \forall \tilde{v}_i, v_i, i \\
& \mathbb{E}_{v_{-i}}[q_i(v)v_i - t_i(v)] \geq 0 \quad \forall v_i, i \\
& q_i(a_i^{optout}, a_{-i}) = 0 \quad \forall i, a_{-i} \\
& t_i(a_i^{optout}, a_{-i}) = 0 \quad \forall i, a_{-i}
\end{aligned} \tag{P}$$

3 Optimal Mechanism

3.1 Main Result

In this section, I define the optimal mechanism and discuss its interpretation. Let p be the optimal posted price, so p solves

$$\max_{\hat{p} \in [0,1]} \hat{p}(1 - \prod_i F_i(\hat{p}))$$

The optimal mechanism has the following allocation function:

$$q_i^{pka}(v) = \begin{cases} 1 & \text{if } v_i = \max_j v_j \geq p \\ 0 & \text{otherwise} \end{cases}$$

where the good goes to the bidder with the highest realized valuation as long as it is above p.

The optimal transfer function collects exactly p in total revenue whenever the good is sold:

$$\sum_i t_i^{pka}(v) = \begin{cases} p & \text{if } \max_i v_i \geq p \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

To give the full specification of the transfer function, it is necessary to first specify the surplus given to each bidder's lowest type. Using incentive compatibility and integration by parts, we can check that the sum of the surpluses given to each bidder's lowest type, $\sum_i U_i(0)$, is weakly positive:

$$\begin{aligned} \sum_i U_i(0) &= \mathbb{E}_v[\sum_i q_i^{pka}(v)v_i - \sum_i t_i^{pka}(v)] - \sum_i \mathbb{E}_{v_i}[\int_0^{v_i} \bar{q}_i^{pka}(\tilde{v})d\tilde{v}] \\ &= \sum_i \mathbb{E}_{v_i}[\mathbb{I}\{v_i \geq p\} \int_p^{v_i} (v_i - p) d \prod_{j \neq i} F_j(\tilde{v})] \geq 0 \end{aligned}$$

As a result, it is possible to divide this surplus among the lowest types so that $U_i(0) \geq 0$ for each i.

In order to satisfy incentive compatibility, the principal's optimal transfer function must yield the following interim transfers while also satisfying 1:

$$\bar{t}_i^{pka}(v_i) = \bar{q}^{pka}(v_i)v_i - U_i(0) - \int_0^{v_i} \bar{q}^{pka}(\tilde{v})d\tilde{v}$$

To accomplish this, define constants κ_i for each i as follows:

$$\kappa_i = \frac{\mathbb{E}_{v_i}[\bar{t}_i^*(v_i)]}{Pr(\max v \geq p)p}$$

κ_i is the proportion of the expected cost paid by bidder i, so $\sum_i \kappa_i = 1$.

We can now construct the principal's optimal transfer function:

$$t_i^{pka}(v) = \kappa_i p\mathbb{I}\{\max v \geq p\} + \bar{t}_i^{pka}(v_i) - \kappa_i \mathbb{E}_{v_{-i}}[p\mathbb{I}\{\max v \geq p\}] - \frac{1}{N-1}\Big(\sum_{j\neq i} \bar{t}_j^{pka}(v_j) - \kappa_j \mathbb{E}_{v_{-j}}[p\mathbb{I}\{\max v \geq p\}]\Big)$$

This direct mechanism (q^{pka}, t^{pka}) satisfies IR, so we can meet the requirement of a safe opt-out action for each bidder by adding actions $\{a_i^{optout}\}_i$ without affecting the truth-telling BNE. We set $q_j^{pka}(a_i^{optout}, a_{-i}) = 0$ and $t_j^{pka}(a_i^{optout}, a_{-i}) = 0$ for all j and i to complete the description of the mechanism. Abusing notation, we refer to the mechanism with the opt-out actions also as (q^{pka}, t^{pka}).

Theorem 1. *The principal's optimal mechanism is* (q^{pka}, t^{pka})*.*

The principal's optimal mechanism collects p in total revenue whenever the good is sold, much like a standard posted price. Unlike the standard posted price, it also manages to allocate the good to the bidder with the highest realized valuation whenever it is above p.[1] To allocate the good efficiently while also capping the total ex post revenue collected at p, the principal uses ex post budget-balanced side payments between the bidders in the mechanism. These side payments can be interpreted as the payments that would implement an efficient pre-auction knockout to determine which bidder gets the right to buy the good at the price p.

3.2 Proof Sketch

In this subsection, I discuss how to show that (q^{pka}, t^{pka}) is optimal. The proof proceeds in four steps, of which I give an overview here. The appendix A contains the details.

3.2.1 Consolidation of EAIR Constraints

First, bidders' EAIR constraints can be combined into a single constraint. Fix an arbitrary mechanism (q, t) that may be set by the principal. Since each bidder's expected surplus from

[1] In general, there is no equilibrium possible with a posted price p that will allocate the good efficiently whenever the maximum realized valuation exceeds p. For this discussion, a posted price p defines a game where only the winner of the good pays, and he pays exactly p. These features constrain the interim transfers possible in any BNE of a posted price, so bidder i with valuation v_i will pay $Prob(v_i \text{ gets the good})p$ in expectation. If $Prob(v_i \text{ gets the good}) = Prob(v_i \geq v_j \forall j \neq i)$ whenever $v_i \geq p$, then the envelope theorem payoff formula constrains the interim allocation function, and it is possible to find examples of distributions where the resulting interim allocation is not consistent with efficient allocation of the good. $F_i = U[0,1]$ for all i is a simple example. See section 4.2 for further discussion of the mechanism with interim transfer function $Prob(v_i \geq v_j \forall j \neq i)\mathbb{I}\{v_i \geq v_t\}$ for some threshold v_t.

collusion must exceed their surplus from playing the truth-telling BNE, the joint surplus of all bidders colluding must exceed the joint non-collusive surplus:

$$\sum_i \mathbb{E}_{v_{-i}}[\tilde{q}_i(v)v_i - \tilde{t}_i(v) - \sum_{\hat{v}} \rho(\hat{v}|v)t_i(\hat{v})] \geq \sum_i U_i^n \tag{Joint EAIR}$$

With ex ante budget-balanced transfers, this is also sufficient for EAIR:

Lemma 1. *Colluders' problem is equivalent to*

$$\inf_{\tilde{q},\tilde{t},\rho} \mathbb{E}_v[\sum_{\hat{v}} \rho(\hat{v}|v) \sum_i t_i(\hat{v})]$$

s.t. Feas, EABB, Joint EAIR, C-IC

3.2.2 Decomposition into BNE Surplus and Cost Function

Next, we can break down the effect of the principal's mechanism into two distinct channels: (1) how it limits the aggregate outcomes that colluders can achieve, and (2) how it shapes bidders' incentives to collude over playing non-collusively. I formalize these ideas in this subsection.

While each action profile of the principal's mechanism specifies the assignment of the good to particular bidders with some probabilities and extracts transfers from individual bidders, these details of an action's outcome can be undermined because colluders can reallocate the good and make payments between each other. For each action profile a in the principal's mechanism, define $(\sum_i q_i(a), \sum_i t_i(a))$ to be the *aggregate outcome* associated with a. As a first step, define

$$A^{(q,t)} = \{(\sum_i q_i(a), \sum_i t_i(a)) : a \in V\},$$

which is the set of aggregate outcomes that can be achieved by playing a single action profile in (q,t) with probability 1. Since colluders can randomize between action profiles, anything in the convex hull of $A^{(q,t)}$ can be induced as an aggregate outcome. Furthermore, colluders can throw away the good, so any aggregate outcome in the following set can be achieved by the cartel:

$$P_{(q,t)} = \{(Q,T) : \exists(\tilde{Q},\tilde{T}) \in conv(A^{(q,t)}) \text{ s.t. } Q \leq \tilde{Q}, T = \tilde{T}, Q \geq 0\}$$

$P_{(q,t)}$ captures the first channel described at the start of this subsection. The second channel is captured by the joint surplus generated by the truth-telling BNE of a mechanism.

Lemma 2. *Fix two mechanisms (q,t) and $(\hat{q},\hat{t})$. Let $\sum_i U_i^n$ $(\sum_i \hat{U}_i^n)$ be the joint surplus associated with the truth-telling BNE under (q,t) $\left((\hat{q},\hat{t})\right)$. If $P_{(q,t)} = P_{(\hat{q},\hat{t})}$ and $\sum_i U_i^n \leq \sum_i \hat{U}_i^n$, then $R_c(q,t) \leq R_c(\hat{q},\hat{t})$.*

The lemma implies that if $P_{(q,t)} = P_{(\hat{q},\hat{t})}$ and $\sum_i U_i^n = \sum_i \hat{U}_i^n$, then $R_c(q,t) = R_c(\hat{q},\hat{t})$. In this way, $P_{(q,t)}$ and $\sum_i U_i^n$ contain all the information about (q,t) needed to pin down the worst-case revenue that could arise. We could think of the principal's problem as equivalent to picking the optimal pair of $P_{(q,t)}$ and $\sum_i U_i^n$.

Working with the space of feasible aggregate outcomes, $P_{(q,t)}$, is cumbersome. Instead, define f to be the lower convex envelope weakly below $cl(P_{(q,t)})$ with domain $[0, \sup_v \sum_i q_i(v)]$. For the rest of the analysis, I work with f instead of $P_{(q,t)}$. I call f the *cost function induced by* (q,t). This object was first defined and studied in Pavlov (2008) and Che and Kim (2009). For any probability Q, $f(Q)$ is the infimal total cost that colluders must pay to get the good with probability Q. f has some important properties, some of which are typical of cost functions:

Lemma 3. *f is (1) continuous on $[0,\bar{q})$, (2) convex, and (3) increasing. Additionally, (4) $f(0) \leq 0$.*

I refer to any function $f : [0,\bar{q}] \to \mathbb{R}$ such that $\bar{q} \in [0,1]$ and f satisfies the properties of Lemma 3 as a *cost function*.

Figure 1 shows the space of feasible aggregate outcomes for a first-price auction with a "strict" reserve price of 0.5. This auction allocates the good to the bidder with the highest bid as long as that bid is strictly above 0.5. Otherwise, no bidder gets the good. $P_{(q,t)}$ is the shaded blue region in the figure. It is bounded below by the function $f(Q) = 0.5Q$ but does not include it. Colluders cannot exactly achieve all points on the cost function, but these points can be approximated arbitrarily closely.

Within the set of mechanisms that induce the same cost function, the principal weakly prefers those that result in strictly higher BNE surplus:

Lemma 4. *Fix mechanisms (q,t) and $(\hat{q},\hat{t})$ with associated BNE surpluses $\sum_i U_i^n$ and $\sum_i \hat{U}_i^n$ respectively. Suppose (q,t) and $(\hat{q},\hat{t})$ induce the same cost function f with domain $[0,\bar{q}]$. If $\sum_i U_i^n > \sum_i \hat{U}_i^n$, then $R_c(q,t) \geq R_c(\hat{q},\hat{t})$.*

3.2.3 Inner Problem: Surplus Maximization

The principal's problem can be split into an inner and an outer problem. The inner problem finds the (direct) mechanism that maximizes BNE surplus among the set of mechanisms that induce a cost function weakly above an exogenously given, continuous cost function

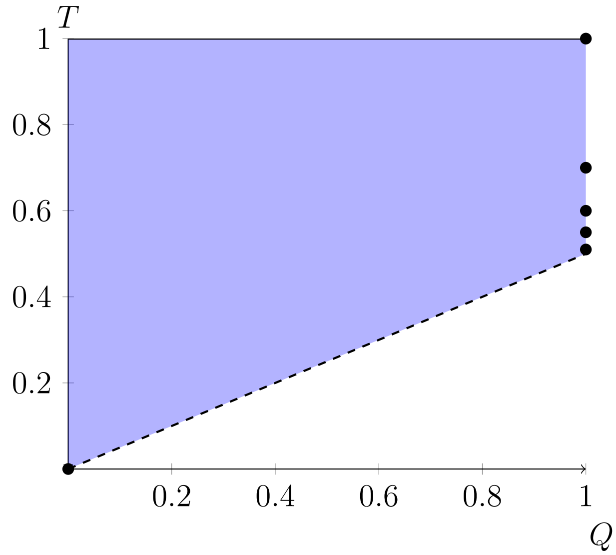

Figure 1: Feasible aggregate outcomes for FPA with a strict reserve price of 0.5 are represented by the blue region. The lower boundary of the set $f(Q) = 0.5Q$ is the cost function and is not included.

$f : [0, \bar{q}] \to \mathbb{R}$. It turns out that the optimal mechanism results in a weakly higher worst-case revenue than any mechanism that induces $\hat{f}$ as its cost function, where $\hat{f}$ equals f except possibly at $\bar{q}$. As a result, it is without loss of optimality for the principal to restrict attention to mechanisms that solve the inner problem for a continuous cost function f. The principal then optimizes over cost functions to find the globally optimal mechanism.

To formulate the principal's inner problem, fix a continuous cost function $f : [0, \bar{q}] \to \mathbb{R}$. Consider the following problem where the feasible (direct) mechanisms induce cost functions weakly above f:

$$\begin{aligned} \max_{q:V\to[0,1]^N, t:V\to\mathbb{R}} \quad & \mathbb{E}_v[\sum_i q_i(v)v_i - t_i(v)] \qquad (f\text{-Surplus Max}) \\ \text{s.t.} \quad & \sum_i q_i(v) \le \bar{q} \quad \forall v \\ & \mathbb{E}_{v_{-i}}[q_i(v)v_i - t_i(v)] \ge \mathbb{E}_{v_{-i}}[q_i(\tilde{v}_i, v_{-i})v_i - t_i(\tilde{v}_i, v_{-i})] \quad \forall v_i, \tilde{v}_i, i \\ & \mathbb{E}_{v_{-i}}[q_i(v)v_i - t_i(v)] \ge 0 \quad \forall i \\ & \sum_i t_i(v) \ge f(\sum_i q_i(v)) \quad \forall v \end{aligned}$$

f-Surplus Max can be solved by considering the relaxed problem without IC and IR constraints:

$$\max_{q:V\to[0,1]^N, t:V\to\mathbb{R}} \quad \mathbb{E}_v[\sum_i q_i(v)v_i - t_i(v)] \tag{Relax fSM}$$
$$\text{s.t.} \quad \sum_i q_i(v) \le \bar{q} \quad \forall v$$
$$\sum_i t_i(v) \ge f(\sum_i q_i(v)) \quad \forall v$$

This relaxed problem can be solved pointwise. The resulting optimal allocation function q^* only allocates the good to the bidder with the highest realized valuation at each v. It acquires the good with a total probability $\sum_i q_i^*(v)$ that depends only on the highest realized valuation. As long as a transfer function t makes the second constraint bind for all v so that $\sum_i t_i(v) = f(\sum_i q_i^*(v))$, (q^*, t) solves the relaxed problem.

It turns out that a particular ex post transfer function t^* makes (q^*, t^*) an IC, IR mechanism:

Proposition 1. *Fix a continuous cost function $f : [0, \bar{q}] \to \mathbb{R}$. There exists t^* such that (q^*, t^*) solves f-Surplus Max and Relax fSM.*

Because the direct mechanism (q^*, t^*) satisfies IR, it is possible to modify it into a mechanism with safe opt-out actions without affecting the BNE. As discussed following Theorem 1, setting $q_j(a_i^{optout}, a_{-i}) = 0$ and $t_j(a_i^{optout}, a_{-i}) = 0$ for all bidders i, j completes the description of the mechanism. Since the truth-telling equilibrium satisfies IR, playing the opt-out action is not a profitable deviation for any type of any agent. From here, I abuse notation and use (q^*, t^*) to also denote the mechanism that combines (q^*, t^*) with opt-out actions and the outcomes defined above.

When the principal sets (q^*, t^*) that solves f-Surplus Max for a continuous cost function f, the worst-case revenue is the revenue from the truth-telling BNE:

Proposition 2. *Suppose (q^*, t^*) solves f-Surplus Max for a continuous cost function f. Then, the value of the colluders' problem facing (q^*, t^*) is $\mathbb{E}_v[f(\sum_i q_i^*(v))]$.*

Given a mechanism (q, t) that induces a cost function $\hat{f} : [0, \bar{q}] \to \mathbb{R}$, it is possible to find a continuous cost function f that equals $\hat{f}$ everywhere except possibly at $\bar{q}$. Given a continuous cost function f, f-Surplus Max is well-defined. The next proposition states that the mechanism (q^*, t^*) that solves f-Surplus Max results is weakly greater worst-case revenue than (q, t):

Proposition 3. *Suppose (q,t) induces cost function $\hat{f}$. Define $f : [0,\bar{q}] \to \mathbb{R}$ so that*

$$f(Q) = \begin{cases} \hat{f}(Q) & \text{if } Q \in [0,\bar{q}) \\ \lim_{Q\to\bar{q}-} \hat{f}(Q) & \text{if } Q = \bar{q} \end{cases}$$

Let (q^*,t^*) solve f-Surplus Max for cost function f. $R_c(q,t) \leq R_c(q^*,t^*)$.

In other words, the principal can improve upon any mechanism (q,t) that results in a cost function $\hat{f}$ by setting the mechanism (q^*,t^*) that solves f-Surplus Max for the related continuous cost function f. Restricting attention to mechanisms that solve f-Surplus Max for a continuous cost function f is without loss of optimality for the principal.

3.2.4 Outer Problem: Optimization over Cost Functions

Given that the principal optimally sets a mechanism that solves f-Surplus Max for a continuous cost function f, the principal's outer problem consists of choosing the optimal such f:

$$\begin{aligned} \max_{\bar{q}\in[0,1], f:[0,\bar{q}]\to\mathbb{R}} \quad & \mathbb{E}_v[f(\sum_i q_i^*(v))] \\ \text{s.t.} \quad & f(0) \leq 0 \\ & (q^*,t^*) \text{ solves } f\text{-Surplus Max} \\ & f \text{ is a continuous cost function} \end{aligned} \tag{P Outer Problem}$$

Recall that the (q^*,t^*) requires the principal to allocate the good to the bidder with the highest realized valuation whenever the good is acquired. Thus, the principal's outer problem is effectively setting a menu of options to allocate the good to a single bidder with valuation equal to $\max_i v_i$, distributed according to $\prod_i F_i$. The menu must be continuous, convex, and increasing, and it must contain the option that the bidder can walk away and receive $-f(0) \geq 0$ in payment. The bidder picks optimally from the menu given his private information, $\max_i v_i$. This problem can be relaxed and reformulated as a standard single-bidder screening problem. From Myerson (1981), we know this has a simple solution: posting a price.

Let p be the optimal posted price so that p solves

$$\max_{\hat{p}\in[0,1]} \hat{p}(1 - \prod_j F_j(\hat{p}))$$

One (complicated) way to implement the posted price is to offer the menu $\{(Q, pQ)\}_{Q\in[0,1]}$. In the terminology of the original framing of P Outer Problem, this corresponds to setting a cost function $f^*(Q) = pQ$ with domain $[0, 1]$. This cost function is continuous, convex, and increasing with $f(0) = 0$, so it is feasible for P Outer Problem and thus solves it. With that, Theorem 1 follows.

4 Extensions

4.1 Surplus Maximizing Colluders

In the main model, the principal evaluates mechanisms according to the worst-case revenue that could arise assuming collusion is restrained only by Feas, C-IC, Joint EAIR, and EABB. Arguably, colluders have a natural interest in maximizing their joint surplus, not in minimizing the principal's revenue, so the principal evaluating mechanisms according to their worst-case revenue may do so with too much pessimism relative to reality. In this subsection, I assume that colluders aim to maximize their joint surplus while accounting for the key frictions discussed in section 2. I do this to determine the extent to which the principal's worst-case evaluation of mechanisms may be driving Theorem 1 as well as to study another appealing model of collusion. Ultimately, I find that the mechanism identified in Theorem 1 remains optimal. In fact, the principal facing surplus-maximizing colluders can maximize revenue just by posting a price.

Instead of evaluating mechanisms according to the worst-case revenue that can arise, the principal first calculates the supremum of colluders' joint surplus that can arise:

$$S(q,t) := \sup_{(\tilde{q},\tilde{t},\rho)} \quad \mathbb{E}_v[\sum_i \tilde{q}_i(v)v_i - \sum_{\tilde{v}} \rho(\tilde{v}|v) \sum_i t_i(\tilde{v})]$$
$$\text{s.t. Feas, C-IC, EABB, Joint EAIR}$$

Because $P_{(q,t)}$ may not necessarily be closed, the supremum may not necessarily be achieved, so we cannot exactly talk about the maximum colluder surplus.

As a result, the principal evaluates mechanism (q, t) according to the following complicated object:

$$\hat{R}_c(q,t) := \lim_{\varepsilon \to 0} \inf \{\mathbb{E}_v[\sum_{\tilde{v}} \rho(\tilde{v}|v) \sum_i t_i(\tilde{v})] : (\tilde{q},\tilde{t},\rho) \text{ satisfies Feas, C-IC, Joint EAIR, EABB and}$$
$$|\mathbb{E}_v[\sum_i \tilde{q}_i(v)v_i - \sum_{\tilde{v}} \rho(\tilde{v}|v) \sum_i t_i(\tilde{v})] - S(q,t)| < \varepsilon\}$$

The principal forms a conservative estimate of his worst-case revenue assuming that colluders implement a collusive scheme that generates surplus arbitrarily close to $S(q, t)$. Now, the principal's problem is

$$\max_{q:A\to[0,1]^N, t:A\to\mathbb{R}^N} \hat{R}_c(q,t)$$

where (q,t) are constrained as in P.

It is possible to once again argue that the principal focuses on choosing between mechanisms that solve f-Surplus Max for a continuous cost function f without loss of optimality. Specifically, I show that the principal weakly prefers the mechanism that solves f-Surplus Max to any mechanism that induces a very similar cost function, $\hat{f}$:

Proposition 4. *Suppose (q,t) induces cost function $\hat{f}$. Define $f:[0,\bar{q}]\to\mathbb{R}$ so that*

$$f(Q) = \begin{cases} \hat{f}(Q) & \text{if } Q \in [0,\bar{q}) \\ \lim_{Q\to\bar{q}-} \hat{f}(Q) & \text{if } Q = \bar{q} \end{cases}$$

Let (q^*,t^*) solve f-Surplus Max for cost function f. $\hat{R}_c(q,t) \leq \hat{R}_c(q^*,t^*)$.

The proof is in Appendix B. Proposition 2 goes through as well, so the rest of the argument proceeds as before. Since the principal can restrict attention to choosing between mechanisms that solve f-Surplus Max, the principal's outer problem is again P Outer Problem. The optimal mechanism identified in Theorem 1 is the principal's optimal mechanism even if colluders are assumed to maximize their joint surplus.

Given colluders coordinate to maximize their joint surplus, simply posting a price of p also maximizes the principal's objective. Surplus-maximizing colluders can run the pre-auction knockout implicit in (p^{pka}, t^{pka}) to allocate the right to buy the good at p to the bidder with the highest valuation above p. Intuitively, the principal no longer has to run the knockout auction in-house because surplus-maximizing bidders will replicate it.

Theorem 2. *The principal's optimal mechanism that maximizes $\hat{R}_c(q,t)$ is posting a price of p.*

4.2 Non-Negative Transfers and Constant Minimum Price

In practice, it is uncommon to observe a principal making payments to bidders in an auction. Such an auction could attract entrants who are not competitive, e.g., always have value 0 for the good, since the optimal auction identified in Theorem 1 makes strictly positive expected payments to some bidders with the lowest valuation.

In this subsection, I consider an extension of the main model where the principal restricts himself to mechanisms where he does not make payments to the bidders. For tractability, I focus on the special case where the principal is restricted to setting a mechanism (q, t) that induces a linear cost function of the form $f(Q) = pQ$ with domain $[0, 1]$ and includes the graph of f in $P_{(q,t)}$. I refer to these mechanisms as *closed and linear*. These mechanisms have a fixed minimum price per unit of probability, which, for example, could be induced by a deterministic reserve price. This class of mechanisms is rich enough to include common mechanisms such as a posted price, a first-price auction with a reserve price, and a second-price auction with a reserve price.

I further assume that all bidders are symmetric so that $F_i = F_j$ for all bidders i and j and F_i satisfies the monotone hazard rate condition:

Assumption 1. *$\frac{1-F_i(v_i)}{f_i(v_i)}$ is decreasing in v_i.*

The principal's optimal mechanism is given by the solution to the following problem for the optimal p:

$$\begin{aligned}
\max_{q:V\to[0,1]^N, t:V\to\mathbb{R}_+} \quad & \mathbb{E}_v[\sum_i q_i(v)v_i - t_i(v)] && \text{(NN-}f\text{-Surplus Max)}\\
\text{s.t.} \quad & \sum_i q_i(v) \leq \bar{q} \quad \forall v \\
& \mathbb{E}_{v_{-i}}[q_i(v)v_i - t_i(v)] \geq \mathbb{E}_{v_{-i}}[q_i(\tilde{v}_i, v_{-i})v_i - t_i(\tilde{v}_i, v_{-i})] \quad \forall v_i, \tilde{v}_i, i \\
& \mathbb{E}_{v_{-i}}[q_i(v)v_i - t_i(v)] \geq 0 \quad \forall i \\
& \sum_i t_i(v) \geq f(\sum_i q_i(v)) \quad \forall v
\end{aligned}$$

The key feature of the optimal mechanism is its ex post transfer function. Consider mechanisms that have transfer functions of the following form:

$$t_i^{\hat{v}_t}(v) = \begin{cases} p & \text{if } v_i = \max_j v_j \geq \hat{v}_t \\ 0 & \text{if o/w} \end{cases}$$

where v_t is exogenous and in $[0, 1]$. In these mechanisms, if there is a bidder with a valuation above the threshold value v_t, the bidder with the highest realized valuation pays p. This ex post transfer function induces an interim transfer function:

$$\bar{t}_i^{\hat{v}_t}(v_i) = \begin{cases} 0 & \text{if } v_i < \hat{v}_t \\ F^{N-1}(v_i)p & \text{if } v_i \geq \hat{v}_t \end{cases}$$

Given incentive compatibility, the interim allocation function is given by the payoff formula, which allows us to write the following differential equation to describe $\bar{q}_i^{\hat{v}_t}(v_i)$ for $v_i \geq \hat{v}_t$:

$$\frac{d\bar{q}_i^{\hat{v}_t}}{dv_i} = \frac{p}{v_i} \underbrace{(N-1)F^{N-2}(v_i)f(v_i)}_{=:\tilde{f}(v_i)}$$

where $\tilde{f}(v_i)$ is defined as the derivative of $F^{N-1}(v_i)$. From here, I also define $\tilde{F}(v_i) := F^{N-1}(v_i)$. $\bar{q}_i^{\hat{v}_t}$ has a positive derivative when $v_i \geq \hat{v}_t$. If there exists an ex post allocation function $q\hat{v}_t$ that induces $\bar{q}_i^{\hat{v}_t}$ as the interim allocation for each i, (q, t) would satisfy incentive compatibility.

Note that for any value of $\hat{v}_t$, the lowest type of each agent must be given 0 surplus since $\bar{t}_i^{\hat{v}_t}(0) = 0$ and no surplus can be given to a type with valuation 0 by giving them the good. As a result, $\bar{q}_i^{\hat{v}_t}(\hat{v}_t)$ must satisfy:

$$\bar{q}_i^{\hat{v}_t}(\hat{v}_t)v_t - \bar{t}_i^{\hat{v}_t}(\hat{v}_t) = 0$$

So, if $v_t \neq 0$, we have that $\bar{q}_i^{\hat{v}_t}(\hat{v}_t) = \frac{\bar{t}_i^{\hat{v}_t}(\hat{v}_t)}{\hat{v}_t}$.

Fixing p, this family of mechanisms is parameterized by $\hat{v}_t$. It turns out that there is a smallest $\hat{v}_t$ at which it is possible to find an ex post allocation function that induces the interim allocation functions specified:

Lemma 5. *Fix a $p \in (0, 1)$. There exists a minimum $\hat{v}_t$ for which there exists an ex post allocation function that induces the interim allocation function $\bar{q}^{\hat{v}_t}$. Denote this minimum by v_t. Additionally, if $v_t > 0$, then*

$$\frac{1 - F(v_t)^N}{N} = \frac{F^{N-1}(v_t)(1 - F(v_t))p}{v_t} + \int_{v_t}^{1} \frac{p}{\tilde{v}} \tilde{f}(\tilde{v})(1 - F(\tilde{v}))d\tilde{v}$$

Under some conditions, the principal's optimal mechanism among closed, linear mechanisms with non-negative transfers is the mechanism with transfers given by t^{v_t} and interim allocation $\bar{q}^{v_t}$, where v_t is as defined in the lemma above.

Proposition 5. *Suppose for all $p \in (0, 1)$, v_t, as described in Lemma 5, is interior.*

Then, the principal's optimal closed and linear mechanism (q^{nnl}, t^{nnl}) is such that:

- *Ex post transfers are given by:*

$$t_i^{nnl}(v) = \begin{cases} p^{nnl} & \text{if } v_i = \max_j v_j \geq v_t^{nnl} \\ 0 & \text{if } o/w \end{cases} \qquad \forall i, v$$

- *Interim allocation functions are given by:*

$$\bar{q}_i^{nnl}(v_t^{nnl}) = \frac{F^{N-1}(v_t^{nnl})p^{nnl}}{v_t^{nnl}}$$

$$\bar{q}_i^{nnl}(v_i) = \int_{v_t^{nnl}}^{v_i} \frac{p^{nnl}}{\tilde{v}}(N-1)F^{N-2}(\tilde{v})f(\tilde{v})d\tilde{v} + \bar{q}_i(v_t^{nnl}) \quad \forall v_i \in [v_t^{nnl}, 1]$$

where $p^{nnl} \in (0, 1)$.

The proof is in Appendix C and uses duality to establish the claim. I form a Lagrangian and then provide feasible dual multipliers such that the mechanism described in the proposition maximizes the Lagrangian and satisfies complementary slackness. The binding IC and IR constraints turn out to be the local upward IC constraints for all types above a threshold v_t^{IR}, which is weakly below v_t, and a range of IR constraints for types between 0 and v_t^{IR}.

5 Conclusion

This paper identifies the posted price plus a knockout auction as the principal's optimal mechanism in the presence of collusion. By concentrating on the principal's surplus maximization subproblem, I significantly narrow down the set of potential optimal mechanisms. This approach highlights critical features of the optimal mechanism, specifically that it efficiently allocates the good when it is sold.

The main model makes relatively few assumptions about cartel behavior. Future research could further the study of this problem by incorporating additional assumptions. For example, in some situations, there might not be an ex ante stage where bidders are uninformed of their valuations. In these cases, the relevant revealed preference constraint for colluders would be an interim rather than an ex ante constraint. Additionally, the ex ante revealed preference constraint implicitly assumes a model of cartel formation where either all bidders opt into the cartel or at least one bidder objects, causing the arrangement to collapse and all bidders to revert to non-collusive play indefinitely. Capturing a different cartel formation process could provide another promising direction for future research. For example, opting out of the cartel could result in all other bidders forming the cartel and then playing non-cooperatively against the bidder left out.

A Proof of Theorem 1

In this section of the appendix, I provide the full proof of Theorem 1.

A.1 Consolidation of EAIR

As a first step, the EAIR constraints can be consolidated into a single constraint on joint surplus.

A.1.1 Lemma 1

Lemma 1: Colluders' problem is equivalent to

$$\inf_{\tilde{q},\tilde{t},\rho} \mathbb{E}_v[\sum_{\hat{v}} \rho(\hat{v}|v) \sum_i t_i(\hat{v})]$$
$$\text{s.t. Feas, EABB, Joint EAIR, C-IC}$$

Proof. From every collusive scheme $\tilde{q}, \rho, \tilde{t}$ that satisfies Joint EAIR, it is possible to construct a $\hat{\tilde{t}}$ that, paired with $\tilde{q}, \rho$, would maintains EABB,C-IC, and Feas while now satisfying EAIR. Define $U_i^c := \mathbb{E}_{v_{-i}}[\tilde{q}_i(v)v_i - \tilde{t}_i(v) - \sum_{\hat{v}} \rho(\hat{v}|v)t_i(\hat{v})]$.

$\sum_i U_i^c \geq \sum_i U_i^n$, so there exist lump sum transfers $\{T_i\}_i$ such that $U_i^c + T_i \geq U_i^n$ and $\sum_i T_i = 0$. Define $\hat{\tilde{t}}_i(v) = \tilde{t}_i(v) + T_i$. This new transfer function when paired with $\tilde{q}$ and ρ satisfies all the constraints in the claim's problem.

□

A.2 Decomposition to BNE Surplus and Cost Function

The next step of the proof is to formalize the decomposition of the effect of the principal's mechanism into two channels. The first channel is captured by the object $P_{(q,t)}$, which was defined in the main text. $P_{(q,t)}$ is the space of aggregate outcomes that can be achieved by the colluders. The second channel is captured by the joint surplus generated by the non-collusive BNE. This high-level claim is formalized by Lemma 2, which demonstrates that if two mechanisms (q, t) and $(\hat{q}, \hat{t})$ are such that $P_{(q,t)} = P_{(\hat{q},\hat{t})}$ and induce the same BNE surpluses, then $R_c(q, t) = R_c(\hat{q}, \hat{t})$. As a result, $P_{(q,t)}$ and the BNE surplus are "sufficient statistics" for a mechanism.

A.2.1 Lemma 2

Lemma 2: Fix two mechanisms (q,t) and $(\hat{q},\hat{t})$. Let $\sum_i U_i^n$ $(\sum_i \hat{U}_i^n)$ be the joint surplus associated with the truth-telling BNE under (q,t) $\big((\hat{q},\hat{t})\big)$. If $P_{(q,t)} = P_{(\hat{q},\hat{t})}$ and $\sum_i U_i^n \leq \sum_i \hat{U}_i^n$, then $R_c(q,t) \leq R_c(\hat{q},\hat{t})$.

Proof. The proof proceeds by pushing notation around. I show that any collusive scheme feasible against $(\hat{q},\hat{t})$ can be mapped to a collusive scheme that is feasible against (q,t) that produces the same expected revenue for the principal. As a result, $R_c(q,t) \leq R_c(\hat{q},\hat{t})$.

To this end, fix any collusive side mechanism feasible against $(\hat{q},\hat{t})$; denote it by $(\hat{\rho},\hat{\hat{q}},\hat{\hat{t}})$. For each $v \in V$, $\hat{\rho}$ maps to a point in the space of aggregate outcomes so that $(\sum_{a\in A}\hat{\rho}(a|v)\sum_i \hat{q}_i(a), \sum_{a\in A}\hat{\rho}($ $P_{(\hat{q},\hat{t})}$. Since $P_{(\hat{q},\hat{t})} = P_{(q,t)}$, there similarly exists a distribution $\rho(\cdot|v)$ over action profiles such that

$$\left(\sum_{a\in A}\rho(a|v)\sum_i q_i(a), \sum_{a\in A}\rho(a|v)\sum_i t_i(a)\right) = \left(\sum_{a\in A}\hat{\rho}(a|v)\sum_i \hat{q}_i(a), \sum_{a\in A}\hat{\rho}(a|v)\sum_i \hat{t}_i(a)\right)$$

For each $v \in V$, it is possible to construct such a $\rho(\cdot|v)$. From here, consider $(\rho,\hat{\hat{q}},\hat{\hat{t}})$. This collusive scheme results in the same revenue as $(\hat{\rho},\hat{\hat{q}},\hat{\hat{t}})$ does, and it inherits Feas, C-IC, and EABB from $(\hat{\rho},\hat{\hat{q}},\hat{\hat{t}})$. Since $\sum_i U_i^n \leq \sum_i \hat{U}_i^n$, $(\rho,\hat{\hat{q}},\hat{\hat{t}})$ satisfies Joint EAIR as well since $(\rho,\hat{\hat{q}},\hat{\hat{t}})$ generates the same joint surplus for colluders as $(\hat{\rho},\hat{\hat{q}},\hat{\hat{t}})$. Since the total colluder surplus under $(\hat{\rho},\hat{\hat{q}},\hat{\hat{t}})$ is weakly greater than $\sum_i \hat{U}_i^n$, Joint EAIR is satisfied by $(\rho,\hat{\hat{q}},\hat{\hat{t}})$ given mechanism (q,t). $\square$

A.2.2 Lemma 3

Ultimately, $P_{(q,t)}$ is difficult to work with, so it will be convenient to summarize $P_{(q,t)}$ with the cost function generated by (q,t), f. f is defined as the lower boundary of the closure of $P_{(q,t)}$.

Lemma 3: f is (1) continuous on $[0,\bar{q})$, (2) convex, and (3) increasing. Additionally, (4) $f(0) \leq 0$.

Proof. We start with (2), convexity. Fix $(Q,f(Q))$, $(Q',f(Q'))$ in $cl(P_{(q,t)})$. For any $\alpha \in [0,1]$, $\alpha(Q,f(Q)) + (1-\alpha)(Q',f(Q')) \in cl(P_{(q,t)})$ because $cl(P_{(q,t)})$ is also convex as the closure of a convex set. So, we must have that $f(\alpha q + (1-\alpha)q') \leq \alpha f(q) + (1-\alpha)f(q')$ by construction of f.

For (3), increasingness, fix $(Q,f(Q))$, $(Q',f(Q'))$ with $Q' > Q$. Suppose $f(Q') < f(Q)$. Fix $\varepsilon > 0$ such that $\min\{1/2(f(Q)-f(Q')), Q'-Q\} > \varepsilon$. We have that $B_\varepsilon((Q',f(Q')) \cap P_{(q,t)} \neq \emptyset$ since $(Q',f(Q')) \in cl(P_{(q,t)})$. Fix $(\tilde{Q},\tilde{T}) \in B_\varepsilon((Q',f(Q')) \cap P_{(q,t)}$. So, $\tilde{Q} > Q$ and $\tilde{T} <$

$\frac{1}{2}(f(Q)-f(Q'))+f(Q')<f(Q)$. Since colluders can dispose of the good, $(Q,\tilde{T})\in P_{(q,t)}$, so we have that $(Q,\tilde{T})\in P_{(q,t)}$ also. This implies that $\tilde{T}\leq f(Q)$. So f is increasing.

From (2), we get that f is continuous on $(0,\bar{q})$. To show (1), continuity on $[0,\bar{q})$, all that remains is to show that $f(0)=f_-(0)$. Because f is increasing, we have that if $f(0)\neq f_-(0)$, it must be that $f(0)<f_-(0)$. Fix $\varepsilon>0$. Consider the function $g(\alpha):=\alpha f(\varepsilon)+(1-\alpha)f(0)$ where $\alpha\in[0,1]$. Observe that $g(0)=f(0)$ and $g(1)=g(\varepsilon)$. g is a continuous function of α and for any $x\in(f(0),f_-(0))$, there must be some $\hat{\alpha}$ such that $g(\hat{\alpha})=x$ since $f_-(0)\leq f(\varepsilon)$. As a result, we have that $\alpha f(\varepsilon)+(1-\alpha)f(0)<f(\alpha\varepsilon)$ since f is convex. Contradiction. $f(0)=f_-(0)$.
Finally, for (4), $f(0)\leq 0$ holds since every bidder i playing a_i^{optout} results in no one getting the good with positive probability and no one paying or being paid. As a result, $(0,0)\in P_{(q,t)}$, and $f(0)\leq 0$. □

From here, I refer to any function $f:[0,\bar{q}]\to\mathbb{R}$ with $\bar{q}\in[0,1]$ and that satisfies the properties listed in Lemma 3 as a *cost function*.

A.2.3 Lemma 4

The cost function f coarsely captures how the principal's mechanism affects what aggregate outcomes colluders can achieve. The next lemma explores how informative the cost function and BNE surplus are of the principal's preferences over mechanisms. Ultimately, not much is lost by summarizing $P_{(q,t)}$ by its induced cost function.

Lemma 4 is a corollary of the following lemma since it is the special case where $f=\hat{f}$ and $\bar{q}=\hat{\bar{q}}$. The more general result will be useful in the third step of the proof, so I state and prove it instead.

Lemma 6. *Fix mechanisms (q,t) and $(\hat{q},\hat{t})$ with associated BNE surpluses $\sum_i U_i^n$ and $\sum_i \hat{U}_i^n$ respectively. Suppose (q,t) $\big((\hat{q},\hat{t})\big)$ induces the cost function f $(\hat{f})$ with domain $[0,\bar{q}]$ $\big([0,\hat{\bar{q}}]\big)$ where $\hat{\bar{q}}\geq\bar{q}$. Furthermore, $f(Q)\geq\hat{f}(Q)$ for all $Q\in[0,\bar{q})$.*

If $\sum_i U_i^n>\sum_i\hat{U}_i^n$, then $R_c(q,t)\geq R_c(\hat{q},\hat{t})$.

Proof. We start by supposing that $R_c(q,t)<R_c(\hat{q},\hat{t})$. Then, there exists $(\tilde{q},\tilde{t},\rho)$ that is feasible against (q,t) and results in revenue $\mathbb{E}_v[\sum_{a\in A}\rho(a|v)\big(\sum_i t_i(a)\big)]<R_c(\hat{q},\hat{t})$. We will construct a collusive scheme against $(\hat{q},\hat{t})$ that has revenue bounded above by $\alpha\mathbb{E}_v[\sum_a\rho(a|v)\big(\sum_i t_i(\tilde{v})\big)]+\delta$ for arbitrarily small $\delta>0$ and α arbitrarily close to 1. As a result, $R_c(\hat{q},\hat{t})\leq\mathbb{E}_v[\sum_a\rho(a|v)\big(\sum_i t_i(\tilde{v})\big)]$, which is a contradiction. We can then conclude that $R_c(\hat{q},\hat{t})\leq R_c(q,t)$.

As a preliminary move, throughout this discussion, we will mostly work with the total transfers that colluders pay at each v, i.e., the transfers inclusive of side payments to other bidders and payments to the principal. Fixing a collusive arrangement $(\rho, \tilde{q}, \tilde{t})$ against mechanism (q, t), we can define $\tilde{t}_i^{total}(v) := \sum_a \rho(a|v)\big(\sum_i t_i(a)\big) + \tilde{t}_i(v)$, the total amount i pays to the principal and the collusive arrangement when the true profile of valuations is v. Similarly, when side payments in a collusive arrangement are not yet specified, with only the bid coordination plan ρ, specifying $\tilde{t}_i^{total}(v)$ pins down the amount that i in side payments makes to other colluders as $\tilde{t}_i^{total}(v) - \sum_{\tilde{v}} \rho(\tilde{v}|v) \sum_i t_i(\tilde{v})$ given the state is v. So, ρ and total payments $\tilde{t}^{total}$ pin down $\tilde{t}$. The EABB constraint on collusion can then be re-expressed as $\mathbb{E}_v[\sum_i \tilde{t}_i^{total}(v)] = \mathbb{E}_v[\sum_{a \in A} \rho(a|v) \sum_i t_i(\tilde{v})]$.

To start, we "scale down" $(\rho, \tilde{q}, \tilde{t})$. We do this to move $\sum_{\tilde{v}} \rho(\tilde{v}|v) \sum_i q_i(\tilde{v})$ slightly away from $\bar{q}$ for each v. This way we will be able to take advantage of the continuity of $\hat{f}$ on $[0, \bar{q})$. Concretely, we consider a slightly modified collusive scheme where, fixing $\alpha \in (0, 1)$, we define

$$\begin{aligned}
\tilde{q}_i^{scale}(v) &= \alpha \tilde{q}_i(v) \\
\tilde{t}_i^{scale}(v) &= \alpha \tilde{t}_i^{total}(v) \\
\rho^{scale}(\tilde{v}|v) &= \alpha \rho(\tilde{v}|v) \quad \text{ if } \tilde{v} \neq a^{optout} \\
\rho^{scale}(\tilde{v}|v) &= \alpha \rho(\tilde{v}|v) + 1 - \alpha \quad \text{ if } \tilde{v} = a^{optout}
\end{aligned}$$

where $a^{optout} = (a_1^{optout}, ..., a_N^{optout})$.

As a collusive scheme against (q, t), $(\rho^{scale}, \tilde{q}^{scale}, \tilde{t}^{scale})$ satisfies feasibility, colluder IC, and EABB. Let's quickly check this:

- Feasibility:

$$\sum_i \tilde{q}_i^{scale}(v) = \sum_i \alpha \tilde{q}_i(v) \leq \alpha \sum_{\tilde{v}} \rho(\tilde{v}|v) \sum_i q_i(\tilde{v}) = \sum_{\tilde{v}} \rho^{scale}(\tilde{v}|v) \sum_i q_i^{scale}(\tilde{v}) \quad \forall v$$

- Colluder IC: Since $(\rho, \tilde{q}, \tilde{t})$ satisfies Colluder IC, we have that the following constraint holds for all v_i and i:

$$\mathbb{E}_{v_{-i}}[\tilde{q}_i(v)v_i - \tilde{t}_i^{total}(v)] \geq \mathbb{E}_{v_{-i}}[\tilde{q}_i(\hat{v}_i, v_{-i})v_i - \tilde{t}_i^{total}(\hat{v}_i, v_{-i})]$$

Multiplying both sides by α gives that

$$\mathbb{E}_{v_{-i}}[\tilde{q}_i^{scale}(v)v_i - \tilde{t}_i^{scale}(v)] \geq \mathbb{E}_{v_{-i}}[\tilde{q}_i^{scale}(\hat{v}_i, v_{-i})v_i - \tilde{t}_i^{scale}(\hat{v}_i, v_{-i})]$$

- EABB: The original collusive scheme $(\rho, \tilde{q}, \tilde{t})$ gives us that

$$\mathbb{E}_v[\sum_{\tilde{v}} \rho(\tilde{v}|v) \sum_i t_i(\tilde{v})] = \mathbb{E}_v[\sum_i \tilde{t}_i^{total}(v)]$$

  Again, multiplying both sides by α gives us that the scaled collusive scheme also satisfies EABB:

$$\mathbb{E}_v[\sum_{\tilde{v}} \rho^{scale}(\tilde{v}|v) \sum_i t_i(\tilde{v})] = \mathbb{E}_v[\sum_i t_i^{scale}(v)]$$

The scaled collusive scheme achieves joint surplus $\alpha \sum_i U_i^n$. For α sufficiently close to 1, we can get $\sum_i \hat{U}_i^n < \alpha \sum_i U_i^n < \sum_i U_i^n$. Note that if $(\rho^{scale}, \tilde{q}^{scale}, \tilde{t}^{scale})$ were feasible as a collusive scheme in $(\hat{q}, \hat{t})$, then we would be done since $(\rho^{scale}, \tilde{q}^{scale}, \tilde{t}^{scale})$ achieves a revenue of $\alpha \mathbb{E}_v[\sum_{a \in A} \rho(a|v) \sum_i t_i(a)] < R_c(\hat{q}, \hat{t})$. The remainder of the argument shows how we can go from $(\rho^{scale}, \tilde{q}^{scale}, \tilde{t}^{scale})$ to something that is feasible against $(\hat{q}, \hat{t})$ by using the information that $(\hat{q}, \hat{t})$ induces the same cost function as (q, t).

Let's make note of some features of the scaled collusive scheme. For each v, define $Q(v) := \alpha \sum_{a \in A} \rho(a|v) \sum_i q_i(a)$; $Q(v)$ is the expected probability with which colluders acquire the good when their valuations are v under $(\rho^{scale}, \tilde{q}^{scale}, \tilde{t}^{scale})$.

$$D := \mathbb{E}_v\Big[\sum_i \tilde{t}_i^{scale}(v)\Big] - \mathbb{E}_v\big[f(Q(v))\big]$$

$D \geq 0$ by the construction of f. If the graph of f was contained in $P_{(\hat{q}, \hat{t})}$, then bidders could minimize the cost they pay to the principal by coordinating their bids to correspond to points exactly on the cost function. D is the cost savings that would result from this. Let's build an auxiliary transfer function:

$$\tilde{t}_{new,i}^{scale}(v) = \tilde{t}_i^{scale}(v) - \frac{1}{N} D$$

Note that $(\tilde{q}^{scale}, \tilde{t}_{new}^{scale})$ satisfies C-IC because transfers are shifted by a constant for each agent i. Furthermore, if there existed a bid coordination scheme that made $(\tilde{q}^{scale}, \tilde{t}_{new}^{scale})$ a feasible collusive scheme, this side mechanism would result in joint surplus $\alpha \sum_i U_i^n + D$.

Fix $\delta > 0$ so that $\alpha \sum_i U_i^n - \sum_i \hat{U}_i^n > \delta$. To complete the construction of our desired collusive scheme, we endeavor to find a point $(Q,T) \in P_{(\hat{q},\hat{t})}$ for each v such that $Q(v) = Q$ and $T < \hat{f}(Q(v)) + \delta$. First, if $\Big(Q(v), \hat{f}\big(Q(v)\big)\Big) \in P_{\hat{q},\hat{t}}$, then there exists $\beta \in \Delta(A)$ such that $\sum_{a\in A} \beta(a)\big(\sum_i \hat{q}_i(a), \sum_i \hat{t}_i(a)\big) = \big(Q(v), \hat{f}(Q(v))\big)$. Set $\hat{\rho}(a|v) = \beta(a)$ for all $a \in A$.

Otherwise, if $\Big(Q(v), \hat{f}\big(Q(v)\big)\Big) \notin P_{(\hat{q},\hat{t})}$, then to find our desired (Q,T), note that $Q(v) < \bar{q}$. Since $\hat{f}$ is continuous on $[0, \bar{q})$, we can find a $\gamma \in \Big(0, \bar{q} - Q(v)\Big)$ such that $|\hat{f}(\tilde{Q}) - \hat{f}(Q(v))| < \delta/2$ for all $\tilde{Q} \in [Q(v), Q(v)+\gamma]$. Consider $\big(Q(v)+\gamma, \hat{f}(Q(v)+\gamma)\big)$. $\big(Q(v)+\gamma, \hat{f}(Q(v)+\gamma)\big) \in cl(P_{(\hat{q},\hat{t})})$, so there exists $(\tilde{Q}, \tilde{T}) \in B_{min\{\delta/2,\gamma\}}\big(Q(v)+\gamma, \hat{f}(Q(v)+\gamma))\big) \cap P_{(\hat{q},\hat{t})}$. Note that $\tilde{Q} > Q(v)$ and $\tilde{T} < \min\{\delta/2, \gamma\} + \delta/2 + \hat{f}(Q(v)) < \hat{f}(Q(v)) + \delta$. So, $(Q(v), \tilde{T}) \in P_{(\hat{q},\hat{t})}$ and there exists $\hat{\rho}(\cdot|v) \in \Delta A$ such that $\sum_a \hat{\rho}(a|v) \sum_i \hat{q}_i(\tilde{v}) \geq Q(v)$ and $\sum_a \hat{\rho}(a|v) \sum_i \hat{t}_i(\tilde{v}) = \tilde{T}$. Fix this $\hat{\rho}(\cdot|v)$.

Finally, we can construct the collusive scheme against $(\hat{q}, \hat{t})$ as desired. Fix the allocation function $\hat{\hat{q}}_i(v) = \tilde{q}_i^{scale}(v)$ for all v, i. Set $\hat{\rho}$ as constructed. We need to adjust the transfer function slightly in order to achieve EABB.

$$B := \mathbb{E}_v[\sum_a \hat{\rho}(a|v) \sum_i \hat{t}_i(a)] - \mathbb{E}_v[\hat{f}(Q(v))] \geq 0$$

Note that $B < \delta$. Define the transfer function to be

$$\hat{\hat{t}}_i(v) = \tilde{t}_{new,i}^{scale}(v) + \frac{1}{N} B$$

$(\hat{\rho}, \hat{\hat{q}}, \hat{\hat{t}})$ is feasible, satisfies colluder IC because it has the same allocation as $\tilde{q}$ and total transfers $\hat{\hat{t}}$ are equal to $\tilde{t}$ plus some constants, EABB by the construction of the transfers, and satisfies Joint EAIR because the surplus from this collusive scheme is at least $\alpha \sum_i U_i^n - \delta$. Meanwhile, the revenue from this collusive arrangement is at most $\alpha \mathbb{E}_v[\sum_a \rho(a|v) \sum_i t_i(a)] + \delta$. We have that

$$\begin{aligned} R_c(\hat{q}, \hat{t}) &\leq \mathbb{E}_v[\sum_a \hat{\rho}(a|v) \sum_i \hat{t}_i(a)] \\ &\leq \alpha \mathbb{E}_v[\sum_a \rho(a|v) \sum_i t_i(a)] + \delta \end{aligned}$$

Since δ could be chosen to be arbitrarily small and α could be chosen to be arbitrarily close to 1, we conclude that $R_c(\hat{q}, \hat{t}) \leq \mathbb{E}_v[\sum_a \rho(a|v) \sum_i t_i(a)]$. This concludes the proof. □

A.3 Inner Problem: Surplus Maximization

In this subsection, I formulate the principal's inner problem. Fix an exogenously given cost function f, which is continuous at $\bar{q}$. The inner problem f-Surplus Max asks what direct mechanism maximizes the surplus generated in non-collusive play subject to inducing a cost function weakly above f.

To solve f-Surplus Max, I relax the problem, discarding the IC and IR constraints, to produce Relax fSM. As alluded to in the main text, we can then solve Relax fSM by solving the pointwise problems:

$$\begin{aligned} \max_{q \geq 0} & \sum_i q_i \cdot v_i - \sum_i t_i \\ \text{s.t.} \quad & \sum_i q_i \leq \bar{q} \\ & \sum_i t_i \geq f(\sum_i q_i) \end{aligned} \tag{v-Problem}$$

It is clear that the (q^*, t^*) that solves v-Problem for a fixed v makes the second constraint bind so that $\sum_i t^* = f(\sum_i q_i^*)$. Furthermore, the good should only be given to the bidder with the highest realized valuation, so given that i is the bidder with the unique highest realized valuation, $q_i^* = \sum_i q^*$. After imposing these two observations, solving the pointwise problem is tantamount to choosing the total probability that the good is given to the bidders, $\sum_j q_j^* =: Q$, for each v:

$$\max_{Q \in [0,\bar{q}]} Q(\max_i v_i) \max_i v_i - f(Q) \tag{reduced-vProb}$$

The objective is continuous, so the problem is well-defined for all $\max_i v_i \in [0, 1]$. Since the objective is a concave function, we have that conditions for local optimality are sufficient for global optimality, and so an optimal $Q^*(\max_i v_i)$ is chosen to satisfy:

$$f'_-(Q^*(\max_i v_i)) \leq \max_i v_i \leq f'_+(Q^*(\max_i v_i))$$

where f'_- and f'_+ define the left and right hand derivatives of f.[2]

Lemma 7. *Suppose cost function $f : [0, \bar{q}] \to \mathbb{R}$ is continuous at $\bar{q}$. The solution to reduced-vProb has the following properties:*

1. *Any selection of maximizers $Q^*(\cdot)$ is non-decreasing.*

[2] Define $f'_-(0) = -\infty$ and $f'_+(1) = \infty$ for the sake of dealing with the corner solutions.

2. *The value of the problem is continuous in* $\max_i v_i$.

3. $f(Q^*(v))$ *is increasing.*

4. *The selection of maximizers* $Q^*(\max_i v_i) := \max \textit{argmax}_{Q\in[0,\bar{q}]} Q \max_i v_i - f(Q)$ *is well-defined and right continuous on* $(0, 1)$.

Proof. Regarding (1), fix v and $\tilde{v}$ such that $\max_i v_i > \max_i \tilde{v}_i$. Let $Q^*(\cdot)$ be any selection of maximizers. Suppose $Q^*(\max_i v_i) < Q^*(\max_i \tilde{v}_i)$. f is convex so it is differentiable almost everywhere. Fix $\tilde{Q} \in (Q^*(\max_i v_i), Q^*(\max_i \tilde{v}_i))$ such that f is differentiable at $\tilde{Q}$. We have that

$$f'_+(Q^*(\max_i v_i)) \leq f'_+(\tilde{Q}) = f'(\tilde{Q}) = f'_-(\tilde{Q}) \leq f'_-(Q^*(\max_i \tilde{v}_i))$$

By the optimality conditions, we have then that $\max_i v_i \leq \max_i \tilde{v}_i$. So $Q^*(\max_i v_i) \geq Q^*(\max_i \tilde{v}_i)$.

Next, we observe that we can invoke Berge's theorem to get that the correspondence of maximizers is compact-valued and upper hemicontinuous as well as (2).

(3) follows from (1) and since f is increasing.

For (4), $Q^*(\cdot)$ as given in the statement is well-defined because the set of maximizers for any instance of the problem is compact. We show that the particular selection in the statement is right-continuous. Fix any decreasing sequence $\{v_n\}_n$ that converges to q. $\{Q^*(v_n)\}_n$ is decreasing and is bounded below by $Q^*(v)$ due to (1), so it is convergent and $Q^*(v) \leq \lim_{n\to\infty} Q^*(v_n)$. The correspondence of maximizers is upper hemicontinuous, so $Q^*(v) \geq \lim_{n\to\infty} Q^*(v_n)$. This implies $Q^*(v) = \lim_{n\to\infty} Q^*(v_n)$. □

Now, we fix a particular solution to Relax fSM to construct q^*. For each $\max_i v_i \in [0, 1]$, define $Q^*(\max_i v_i) := \max \operatorname{argmax}_{Q\in[0,\bar{q}]} Q \max_i v_i - f(Q)$. Define the following allocation function for all i and v:

$$q_i^*(v) = \begin{cases} \frac{1}{|\{v_j : v_j = \max_k v_k\}|} Q^*(\max_j v_j) & \text{if } v_i = \max_j v_j \\ 0 & \text{otherwise} \end{cases}$$

Given any transfer function t such that $\sum_i t_i(v) = f(Q^*(\max_j v_j))$ for all v, (q^*, t) solves Relax fSM. The next result shows that with a particular choice of transfer function t^*, (q^*, t^*) also solves f-Surplus Max.

A.3.1 Proposition 1

Proposition 1: Fix a continuous cost function $f : [0, \bar{q}] \to \mathbb{R}$. There exists t^* such that (q^*, t^*) solves f-Surplus Max and Relax fSM.

Proof. Observe that $q^*(\cdot)$ induces weakly increasing interim allocation functions since as an bidder's valuation increases (1) the probability that they hold the highest realized valuation increases, and (2) the total probability at which the good will be given to the bidder increases. So, if we can provide a transfer function that satisfies the envelope theorem derived formula for interim transfers and satisfies the ex post restrictions from Relax fSM on total cost paid a.e., we have demonstrated that f-Surplus Max = Relax fSM.

The first step of the proof is to demonstrate that if we could find a transfer function t^* so that (q^*, t^*) is IC, we could also satisfy IR. To do this, notice that given t^* satisfies IC, we can calculate the total surplus given to each bidders' lowest types via the payoff formula:

$$\sum_i U_i(0) = \mathbb{E}_v\left[\sum_i q_i^*(v)v_i - \sum_i t_i^*(v)\right] - \sum_i \mathbb{E}_{v_i}\left[\int_0^{v_i} \bar{q}_i^*(\tilde{v})d\tilde{v}\right]$$

where $\sum_i t_i^*(v) = f(\sum_i q_i^*(v))$ for all v.

Let $\tilde{q}^* : [0,1] \to [0,1]$ be the mapping from the maximum realized valuation to the optimal probability at which to acquire the good. Lemma 7 establishes that $\tilde{q}^*$ is increasing and is right-continuous on $(0,1)$, so it is differentiable almost everywhere with jump discontinuities only at countable locations. Let V_d be the set of discontinuities in $\tilde{q}^*$.

Since $\tilde{q}^*$ is increasing, we can identify a threshold v_t at which q^* starts being strictly positive. Formally, define:

$$v_t := \sup\{v \in [0,1] : Q^*(v) = 0\}$$

For $v > v_t$, we can define $\tilde{f}(v) := f(\tilde{q}^*(v))/\tilde{q}^*(v)$, the optimal cost per unit of probability that colluders will pay in total when the maximum value is v. Note that since f is continuous on $[0, \bar{q}]$, $\tilde{f}$ is discontinuous at $v \in [0,1]$ if and only if $\tilde{q}$ is discontinuous at v.

Let's massage the expression for $\sum_i U_i(0)$:

$$
\begin{aligned}
\sum_i U_i(0) &= \sum_i \int_{v_t}^1 \left[\bar{q}_i^*(v)(v_i - \tilde{f}(v_i)) - \int_0^{v_i} \bar{q}_i^*(\tilde{v}) d\tilde{v}\right] dF_i(v_i) + \int_0^{v_t} -f(0) dF_i(v_i) \\
&= \sum_i \underbrace{\int_{v_t}^1 \left[(\prod_{j\neq i} F_j(v_i)\tilde{q}_i^*(v_i))(v_i - \tilde{f}(v_i)) - \int_{v_t}^{v_i} \prod_{j\neq i} F_j(\tilde{v})\tilde{q}_i^*(\tilde{v}) d\tilde{v}\right] dF_i(v_i)}_{=:S_i} + \underbrace{\int_0^{v_t} -f(0) dF_i(v_i)}_{\geq 0}
\end{aligned}
$$

The second term is weakly positive since $f(0) \leq 0$, so we concentrate on assigning a sign to S_i.

We can integrate the last term by parts:

$$
\int_{v_t}^{v_i} \prod_{j\neq i} F_j(v_i)\tilde{q}_i^*(v_i) d\tilde{v} = \prod_{j\neq i} F_j(v_i)\tilde{q}^*(v_i)v_i - \sum_{v\in V_d \cap (v_t, v_i)} [\prod_{j\neq i} F_j(s)\tilde{q}^*(s)s]_{v-}^{v+} - \int_{v_t}^{v_i} \tilde{v} d\big(\prod_{j\neq i} F_j(\tilde{v})\tilde{q}^*(\tilde{v})\big)
$$

Putting it back together:

$$
\begin{aligned}
S_i = &\int_{v_t}^1 \Bigg[(\prod_{j\neq i} F_j(v_i)\tilde{q}_i^*(v_i))(v_i - \tilde{f}(v_i)) - \prod_{j\neq i} F_j(v_i)\tilde{q}^*(v_i)v_i + \sum_{v\in V_d\cap(v_t,v_i)} [\prod_{j\neq i} F_j(s)\tilde{q}^*(s)s]_{v-}^{v+} \\
&+ \int_{v_t}^{v_i} \tilde{v} d\big(\prod_{j\neq i} F_j(\tilde{v})\tilde{q}^*(\tilde{v})\big)\Bigg] dF_i(v_i) \\
= &\int_{v_t}^1 \Bigg[\int_{v_t}^{v_i} \tilde{v} - \tilde{f}(v_i) d\big(\prod_{j\neq i} F_j(\tilde{v})\tilde{q}^*(\tilde{v})\big) + \sum_{v\in V_d\cap(v_t,v_i)} [\prod_{j\neq i} F_j(s)\tilde{q}^*(s)(s - \tilde{f}(v_i))]_{v-}^{v+}\Bigg] dF_i(v_i) \\
= &\int_{v_t}^1 (1 - F_i(v_i))(v_i - \tilde{f}(v_i)) \left(\frac{d\big(\prod_{j\neq i} F_j(\tilde{v})\tilde{q}^*(\tilde{v})\big)}{d\tilde{v}}\right)\Bigg|_{\tilde{v}=v_i} dv_i \\
&+ \sum_{v\in V_d} (1 - F_i(v))[\prod_{j\neq i} F_j(s)\tilde{q}^*(v)(v - \tilde{f}(v))]_{v-}^{v+} \geq 0
\end{aligned}
$$

where $\frac{d\left(\prod_{j\neq i} F_j(\tilde{v})\tilde{q}^*(\tilde{v})\right)}{d\tilde{v}}$ is defined almost everywhere and weakly positive because $\prod_{j\neq i} F_j(\tilde{v})\tilde{q}^*(\tilde{v})$ is increasing. Furthermore, $v - \tilde{f}(v) \geq 0$ for all v since surplus maximization implies that $\tilde{q}(v)v - f(\tilde{q}(v)) \geq 0$.

If $\mathbb{E}_v[f(\sum_i q_i^*(v))] \neq 0$, we can take any division of the surplus to the lowest types such that for all i, $U_i(0) \geq 0$. We will show that the value of Relax fSM can be achieved by constructing a transfer function t^* that satisfies the payoff formula and hits the ex post constraints required for optimality identified by the solution to Relax fSM. To this end, define constants κ_i so that

$$\kappa_i = \frac{\mathbb{E}_{v_i}[\bar{t}_i^*(v_i)]}{\mathbb{E}_v[f(\sum_i q_i(v))]}$$

where $\bar{t}_i^*(v_i)$ is given by the payoff formula

$$\bar{t}_i^*(v_i) = \bar{q}^*(v_i)v_i - U_i(0) - \int_0^{v_i} \bar{q}^*(\tilde{v})d\tilde{v}$$

The following transfers satisfy the constraints $\sum_i t_i^*(v) = f(\sum_i q_i^*(v))$ for all v and match the interim transfers required for IC:

$$\begin{aligned} t_i(v) = \kappa_i f(\sum_k q_k^*(v)) + \bar{t}_i^*(v_i) - \kappa_i \mathbb{E}_{v_{-i}}[f(\sum_k q_k^*(v))] \\ - \frac{1}{N-1}\big(\sum_{j\neq i} \bar{t}_j^*(v_j) - \kappa_j \mathbb{E}_{v_{-j}}[f(\sum_k q_k^*(v))]\big) \end{aligned}$$

If $\mathbb{E}_v[f(\sum_i q_i^*(v))] = 0$, consider a particular restriction on the division of surplus between the lowest types. The idea is to also restrict each bidder's expected transfers to be 0, i.e.,

$$\mathbb{E}_{v_i}[\bar{t}_i^*(v_i)] = 0 \quad \forall i$$

This restriction pins down the surplus to the lowest type for each bidder through the payoff formula:

$$U_i(0) = \mathbb{E}_{v_i}[\bar{q}_i^*(v_i)v_i - \int_0^{v_i} \bar{q}_i^*(\tilde{v})d\tilde{v}] \geq 0$$

where the inequality holds because $\bar{q}^*(\cdot)$ is weakly increasing in v_i. Now, the following transfer rule satisfies the desired interim transfers and $\sum_i t_i^*(v) = f(\sum_k q_k^*(v))$ for all v:

$$\begin{aligned} t_i^*(v) = \frac{1}{N} f(\sum_k q_k^*(v)) + \bar{t}_i^*(v_i) - \frac{1}{N}\mathbb{E}_{v_{-i}}[f(\sum_k q_k^*(v))] \\ - \frac{1}{N-1}\Bigg(\sum_{j\neq i} \bar{t}_j^*(v_j) - \frac{1}{N}\mathbb{E}_{v_{-j}} f(\sum_k q_k^*(v))]\Bigg) \end{aligned}$$

This concludes the proof. $\square$

The mechanism that solves f-Surplus Max lacks opt-out actions for each bidder, but since it satisfies IR, it is easy to remedy this by adding actions. As described in the main text, I augment (q^*, t^*) by defining $q_j^*(a_i^{optout}, a_{-i}) = 0$ and $t_j^*(a_i^{optout}, a_{-i})$ for all i, j, and

a_{-i}. IR ensures that no bidder will deviate from their strategy in the truth-telling BNE to playing their opt-out action. This augmented mechanism is feasible for the principal. From here, I use (q^*, t^*) to refer to the solution of f-Surplus Max augmented with opt-out actions as described.

A.3.2 Proposition 2

Ultimately, I will show that it is without loss for the principal to restrict attention to optimizing over mechanisms that solve f-Surplus Max for a continuous cost function f. To that end, we should know what the worst-case revenue generated by these mechanisms is.

Proposition 2: Suppose (q^*, t^*) solves f-Surplus Max for a continuous cost function f. Then, the value of the colluders' problem facing (q^*, t^*) is $\mathbb{E}_v[f(\sum_i q_i^*(v))]$.

Proof. Facing (q^*, t^*), colluders' adversarial collusion results in at most the revenue from the principal's preferred BNE, $\mathbb{E}_v[f(\sum_i q_i^*(v))]$. I show that $R_c(q^*, t^*) = \mathbb{E}_v[f(\sum_i q_i^*(v))]$ by showing that every collusive scheme results against (q^*, t^*) must result in revenue at least $\mathbb{E}_v[f(\sum_i q_i^*(v))]$.

First, observe that with probability 1, there is a single bidder with the maximum realized valuation. Next, let

$$V_m := \{v \in [0, 1] : \mathit{reduced - vProb} \text{ has at least two solutions}\}$$

With probability 1, the solution to reduced-vProb is unique:

Lemma 8. *V_m has measure* 0.

Proof. Fix a $\max_i v_i \in V_m$. Let Q and Q' be two distinct solutions to reduced-vProb with $Q < Q'$. The concavity of the objective implies that any convex combination of Q and Q' also solves the problem, so $[Q, Q']$ is a subset of the solutions to reduced-vProb. The local optimality conditions imply that $f'(\tilde{Q}) = \max_i v_i$ for all $\tilde{Q} \in (Q, Q')$. Denote the interval (Q, Q') by $I(\max_i v_i)$. Every $\max_i v_i \in V_m$ can be mapped in this way to a non-degenerate $I(\max_i v_i)$. These intervals are pairwise disjoint, i.e. $I(\max_i v_i) \cap I(\max_i \tilde{v}_i) = \emptyset$, since $Q \in I(\max_i v_i) \cap I(\max_i \tilde{v}_i)$ implies $\max_i v_i = f'(Q) = \max_i \tilde{v}_i$. We can map each $\max_i v_i \in V_{max,m}$ to a distinct rational $q \in I(\max_i v_i) \cap \mathbb{Q}$. Thus, $V_{max,m}$ is countable and so has measure 0. $\square$

Fix a collusive scheme against (q^*, t^*), $(\tilde{q}, \tilde{t}, \rho)$. The special feature of (q^*, t^*) is that it maximizes joint total surplus for each v given a cost function $f(\cdot)$:

$$\begin{aligned}\sum_i q_i^*(v)v_i - t_i^*(v) &= Q^*(\max_i v_i)\max_i v_i - f(Q^*(\max_i v_i)) \\ &\geq \sum_i \tilde{q}_i(v)v_i - \sum_{\tilde{v}} \rho(\tilde{v}|v)\sum_i t_i^*(\tilde{v})\end{aligned}$$

In the second line, I observe that the collusive scheme's effective contribution to joint surplus is weakly below the joint surplus generated by the BNE at each v. Since any collusive scheme $(\tilde{q}, \tilde{t}, \rho)$ against (q^*, t^*) must raise at least $\mathbb{E}_v[\sum_i \tilde{q}_i^*(v)v_i - f(\sum_i \tilde{q}_i^*(v))]$ in surplus for the colluders, the above inequality must bind with probability 1. With probability 1, there is a unique bidder with the highest realized valuation, and by Lemma 8, we have that $\sum_i \tilde{q}_i(v) = Q^*(\max_i v_i)$ with probability 1. Let

$$\begin{aligned}A := \{v \in V : \sum_i q_i^*(v)v_i - t_i^*(v) = \sum_i \tilde{q}_i(v)v_i - \sum_{\tilde{v}} \rho(\tilde{v}|v)\sum_i t_i^*(\tilde{v}) \\ \text{and reduced-vProb has a unique solution}\}\end{aligned}$$

Note that A occurs with probability 1. For each $v \in A$, we have that $\sum_{\tilde{v}} \rho(\tilde{v}|v)\sum_i t_i^*(\tilde{v}) \geq f(Q^*(\max_i v_i))$. So we have that $\mathbb{E}_v[\sum_{\tilde{v}} \rho(\tilde{v}|v)\sum_i t_i^*(\tilde{v})] \geq \mathbb{E}_v[f(\sum_i Q^*(\max_i v_i))]$. $\square$

A.3.3 Proposition 3

Fix a cost function $\hat{f} : [0, \bar{q}] \to \mathbb{R}$. Define $f : [0, \bar{q}] \to \mathbb{R}$ so that $f(Q) = \hat{f}(Q)$ for all $Q \in [0, \bar{q})$ and $f(\bar{q}) = \hat{f}_-(\bar{q})$. This cost function is nearly $\hat{f}$ but is continuous at $\bar{q}$. It will lead to a well-defined f-Surplus Max.

Given the value of setting a mechanism that solves f-Surplus Max for cost function f, we can now show that any mechanism (q, t) that induces the cost function $\hat{f}$ achieves a weakly lower worst-case revenue than (q^*, t^*).

Proposition 3: Suppose (q, t) induces cost function $\hat{f}$. Define $f : [0, \bar{q}] \to \mathbb{R}$ so that

$$f(Q) = \begin{cases} \hat{f}(Q) & \text{if } Q \in [0, \bar{q}) \\ \lim_{Q \to \bar{q}-} \hat{f}(Q) & \text{if } Q = \bar{q} \end{cases}$$

Let (q^*, t^*) solve f-Surplus Max for cost function f. $R_c(q, t) \leq R_c(q^*, t^*)$.

Proof. From Proposition 2, $R_c(q^*, t^*) = \mathbb{E}_v[f(\sum_i q_i^*(v))]$.

To show that $R_c(q,t)$ is weakly below $R_c(q^*, t^*)$, let $\sum_i U_i^n$ be the surplus from the truth-telling BNE of (q,t), and let $\sum_i U_i^*$ be the surplus from (q^*, t^*). Since (q^*, t^*) solves f-Surplus Max and $f \leq \hat{f}$, we have that $\sum_i U_i^n \leq \sum_i U_i^*$. Consider two cases:

- $\sum_i U_i^n < \sum_i U_i^*$. Lemma 6 applies.
- $\sum_i U_i^n = \sum_i U_i^*$. First, observe that the following inequality holds for all v:

$$\begin{aligned}\sum_i q_i(v)v_i - \sum_i t_i(v) &\leq \sum_i q_i(v) \max_j v_j - \hat{f}(\sum_i q_i(v)) \\ &\leq Q^*(\max_i v_i) \max_i v_i - f(Q^*(\max_i v_i))\end{aligned}$$

  since $\hat{f} \geq f$. Furthermore, this inequality must bind for almost all v since $\sum_i U_i^n = \sum_i U_i^*$. Given that $\hat{f}(\bar{q}) > f(\bar{q})$, then the set $\{v \in V : Q^*(v) = \bar{q}\}$ must have probability 0. So, with probability 1, $Q^*(v) < \bar{q}$. The above inequalities holding at equality almost everywhere allows us to use Lemma 8 to conclude that for almost all v, $Q^*(v) = \sum_i q_i(v)$. As a result, the revenue from the non-collusive BNE of (q,t) is equal to the revenue from the non-collusive BNE of (q^*, t^*):

$$\mathbb{E}_v[\sum_i t_i(v)] = \mathbb{E}_v[f(Q^*(v))]$$

  We have that $R_c(q,t) \leq \mathbb{E}_v[f(Q^*(v))] \leq R_c(q^*, t^*)$.

□

A.4 Outer Problem: Optimization of Cost Function

The previous steps of the argument show that it is without loss for the principal to optimize over mechanisms that solve f-Surplus Max for a continuous cost function f. In this section, the principal's problem is completed with an outer optimization over continuous cost functions, P Outer Problem.

Since (q^*, t^*) that solves f-Surplus Max allocates the good to the bidder with the highest realized valuation, we can also write the principal's problem as

$$\begin{aligned}
\max_{\bar{q}\in[0,1],f:[0,\bar{q}]\to\mathbb{R}} \quad & \mathbb{E}_v[f(Q^*(v))] \\
\text{s.t.} \quad & f(0) \leq 0 \\
& Q^*(\max_i v_i) \text{ solves reduced-vProb} \\
& f \text{ is a continuous cost function}
\end{aligned}$$

This problem can further be relaxed to the following problem:

$$\begin{aligned}
\max_{\bar{q}\in[0,1],f:[0,\bar{q}]\to\mathbb{R}} \quad & \mathbb{E}_v[f(Q^*(\max_i v_i))] \\
\text{s.t.} \quad & Q^*(\max_i v_i) \text{ solves reduced-vProb} \\
& \text{reduced-vProb} \geq 0 \quad \forall \max_i v_i
\end{aligned}$$

where the requirements that f be convex, increasing, and continuous have been dropped and $f(0) \leq 0$ has been replaced by the milder necessary condition that the value of v-Problem be weakly greater than 0 for each v[3]. Using the revelation principle, this relaxed problem is equivalent to designing a revenue-maximizing direct mechanism to sell the good to a single bidder subject to the usual incentive compatibility and individual rationality conditions.

Using Myerson (1981), the optimal mechanism to set as a revenue-maximizing principal facing a single bidder is to post a price. Let p be the optimal price. As discussed in the main text, this mechanism can be implemented with a cost function $f(Q) = pQ$. The optimal mechanism is such that the truth-telling BNE solves f-Surplus Max with $f(Q) = pQ$, with opt-out actions included as described in the main text. This is the mechanism discussed in Section 3.1.

[3]Requiring a convex and increasing f is without loss for P Outer Problem. Suppose $Q' \leq Q$ but $f(Q') > f(Q)$. Then, by choosing $(Q, f(Q))$ over $(Q', f(Q'))$, the bidder gets the good with at least as much probability for strictly less cost. The optimizing bidder would never choose $(Q', f(Q'))$ in equilibrium; $(Q', f(Q'))$ could be replaced by $(Q', f(Q))$ with no impact on the principal's revenue. In this sense, requiring f to be increasing is without loss. It is a result of the bidder's optimizing behavior, not an exogenous restriction on the types of menus that can be set. Similarly, a convex f is without loss if the bidder being screened can play mixed strategies.

B Proposition 4

Proposition 4: Suppose (q, t) induces cost function $\hat{f}$. Define $f : [0, \bar{q}] \to \mathbb{R}$ so that

$$f(Q) = \begin{cases} \hat{f}(Q) & \text{if } Q \in [0, \bar{q}) \\ \lim_{Q \to \bar{q}-} \hat{f}(Q) & \text{if } Q = \bar{q} \end{cases}$$

Let (q^*, t^*) solve f-Surplus Max for cost function f. $\hat{R}_c(q, t) \leq \hat{R}_c(q^*, t^*)$.

Proof. As a first step, notice that $\hat{R}_c(q^*, t^*) = \mathbb{E}_v[f(\sum_i q^*(v))]$. This follows from the proof of Proposition 2, which uses Joint EAIR to conclude that any possible collusive arrangement must result in the same expected revenue as the truth-telling BNE of (q^*, t^*). Adding assumptions about colluder behavior does not change this argument.

Next, let $\sum_i U_i^n$ be the BNE surplus of (q, t), and let $\sum_i U_i^*$ be the BNE surplus of (q^*, t^*). I consider two cases: (1) $\sum_i U_i^n < \sum_i U_i^*$ (2) $\sum_i U_i^n = \sum_i U_i^*$. In the second case, the proof from Proposition 3 for this case $\sum_i U_i^n = \sum_i U_i^*$ showed that any collusive scheme that satisfies JEAIR must also produce $\mathbb{E}_v[f(\sum_i q^*(v))]$ in revenue. This observation still applies in this case and pins down $\hat{R}_c(q, t) = \hat{R}_c(q^*, t^*)$. In the first case, $\sum_i U_i^*$ is an upper bound on the total joint surplus that can be generated by collusion against (q, t). Furthermore, in the proof of Lemma 6, I provided a construction of a collusive arrangement against (q, t) that approximates the BNE of (q^*, t^*). This collusive scheme produces surplus of at least $\alpha \sum_i U_i^* - \delta$ where $\delta > 0$ and is arbitrarily small and α is arbitrarily close to 1. This collusive scheme results in revenue which is at most $\mathbb{E}_v[\hat{f}(\sum_i q^*(v))] + \delta$. As a result, $\hat{R}_c(q, t) \leq \hat{R}_c(q^*, t^*)$. $\square$

C Non-Negative Transfers, Linear Cost Functions

C.1 Lemma 5

We directly check the Border inequalities (Border 1991):

$$\int_{v_i}^{1} \bar{q}(\tilde{v}) dF(\tilde{v}) \leq \frac{1 - F(v_i)^N}{N} \quad \forall v_i \in [0, 1]$$

to verify the existence of an ex post allocation that corresponds to the interim allocation function. Note that the expression on the RHS is increasing as v_i decreases since $F(v_i)$

decreases as v_i increases. So it is sufficient to check the above inequality for $v_i \in [\hat{v}_t, 1]$ since satisfying the inequality at $v_i = \hat{v}_t$ implies it is satisfied for all $v_i < \hat{v}_t$.

Let's massage the expression on the LHS, dropping the i subscripts due to the symmetry:

$$\begin{aligned}
\int_v^1 \bar{q}(\tilde{v})dF(\tilde{v}) &= \int_v^1 \big(\int_{\hat{v}_t}^{\tilde{v}} \frac{d\bar{q}(\hat{v})}{d\hat{v}} d\hat{v} + \bar{q}(\hat{v}_t)\big) dF(\tilde{v}) \\
&= \big(\int_{\hat{v}_t}^{\tilde{v}} \frac{p}{\hat{v}} \tilde{f}(\hat{v}) d\hat{v} + \bar{q}(\hat{v}_t)\big)(F(\tilde{v}) - F(v))|_{\tilde{v}=v}^{1} - \int_v^1 \frac{p}{\tilde{v}} \tilde{f}(\tilde{v})(F(\tilde{v}) - F(v))d\tilde{v} \\
&= \left(\int_{\hat{v}_t}^{1} \frac{p}{\hat{v}} \tilde{f}(\hat{v}) d\hat{v} + \bar{q}(\hat{v}_t)\right)(1 - F(v)) - \int_v^1 \frac{p}{\tilde{v}} \tilde{f}(\tilde{v})(F(\tilde{v}) - F(v))d\tilde{v} \\
&= \bar{q}(\hat{v}_t)(1 - F(v)) + \int_{\hat{v}_t}^{v} \frac{p}{\tilde{v}} \tilde{f}(\tilde{v})(1 - F(v))d\tilde{v} + \int_v^1 \frac{p}{\tilde{v}} \tilde{f}(\tilde{v})(1 - F(v))d\tilde{v} \\
&\quad - \int_v^1 \frac{p}{\tilde{v}} \tilde{f}(\tilde{v})(F(\tilde{v}) - F(v))d\tilde{v} \\
&= \bar{q}(\hat{v}_t)(1 - F(v)) + \int_{\hat{v}_t}^{v} \frac{p}{\tilde{v}} \tilde{f}(\tilde{v})(1 - F(v))d\tilde{v} + \int_v^1 \frac{p}{\tilde{v}} \tilde{f}(\tilde{v})(1 - F(\tilde{v}))d\tilde{v}
\end{aligned}$$

Lemma 9. *If $\hat{v} > v > 0$, and the Border inequalities are satisfied with $\hat{v}_t = v$, then the Border inequalities are satisfied with $\hat{v}_t = \hat{v}$.*

Proof. To show this, we focus on how the LHS of the Border inequalities changes as $\hat{v}_t$ changes to demonstrate that as $\hat{v}_t$ increases, the LHS decreases. Recall that $\bar{q}(\hat{v}_t) = \frac{p}{\hat{v}_t} F^{N-1}(\hat{v}_t)$. Let's then calculate the derivative of the LHS wrt $\hat{v}_t$ for $v > \hat{v}_t$:

$$\begin{aligned}
\frac{dLHS_u}{d\hat{v}_t} &= -\frac{p}{\hat{v}_t} \tilde{f}(\hat{v}_t)(1 - F(\hat{v}_t)) + (1 - F(v)) \frac{d\bar{q}(\hat{v}_t)}{d\hat{v}_t} \\
&= -\frac{p}{\hat{v}_t} \tilde{f}(\hat{v}_t)(1 - F(\hat{v}_t)) + (1 - F(v)) \left(\frac{\tilde{f}(\hat{v}_t)p}{\hat{v}_t} - \frac{F^{N-1}(\hat{v}_t)p}{(\hat{v}_t)^2} \right) \\
&= -\frac{p}{\hat{v}_t} \tilde{f}(\hat{v}_t)(F(v) - F(\hat{v}_t)) - (1 - F(v)) \frac{F^{N-1}(\hat{v}_t)p}{(\hat{v}_t)^2} < 0
\end{aligned}$$

As a result, given $\hat{v} > v$ and that the Border inequalities are satisfied with $\hat{v}_t = v$, we have that for all $v \geq \hat{v}$, the change in the LHS going from $\hat{v}_t = v$ to $\hat{v}_t = \hat{v}$ is

$$\int_v^{\hat{v}} \frac{dLHS}{d\hat{v}_t} d\tilde{v} < 0$$

For $\tilde{v} < \hat{v}_t$, the LHS is

$$\bar{q}(\hat{v}_t)(1 - F(\hat{v}_t)) + \int_{\hat{v}_t}^{1} \frac{p}{\tilde{v}} \tilde{f}(\tilde{v})(1 - F(\tilde{v}))d\tilde{v}$$

and the derivative of the LHS is

$$\begin{aligned}\frac{dLHS_l}{d\hat{v}_t} &= -\frac{p}{\hat{v}_t}\tilde{f}(\hat{v}_t)(1-F(\hat{v}_t)) + (1-F(\hat{v}_t))\frac{d\bar{q}(\hat{v}_t)}{d\hat{v}_t} - f(\hat{v}_t)\bar{q}(\hat{v}_t) \\ &= -f(\hat{v}_t)\bar{q}(\hat{v}_t) - (1-F(v))\frac{F^{N-1}(\hat{v}_t)p}{(\hat{v}_t)^2} < 0\end{aligned}$$

For $\tilde{v} \in [v, \hat{v}]$, the change in the LHS is:

$$\int_v^{\tilde{v}} \frac{dLHS_u}{d\hat{v}_t}(\tau)d\tau + \int_{\tilde{v}}^{\hat{v}} \frac{dLHS_l}{d\hat{v}_t}(\tau)d\tau < 0$$

Since the LHS of the Border inequality decreases for every $\tilde{v}$, the Border inequalities continue to hold at $\hat{v}$. □

Lemma 10. *The Border inequalities are satisfied with $\hat{v}_t = p$.*

Proof. Take $\hat{v}_t = p$. As a result, $\bar{q}(p) = F^{N-1}(p)$. The Border inequalities are

$$F^{N-1}(p)(1-F(v)) + \int_p^v \frac{p}{\tilde{v}}\tilde{f}(\tilde{v})(1-F(v))d\tilde{v} + \int_v^1 \frac{p}{\tilde{v}}\tilde{f}(\tilde{v})(1-F(\tilde{v}))d\tilde{v} \overset{?}{\leq} \frac{1-F^N(v)}{N}$$

Let's take the derivative of both sides of the inequality with respect to v.

$$\begin{aligned}\frac{dLHS}{dv} &= -F^{N-1}(p)f(v) - f(v)\int_p^v \frac{p}{\tilde{v}}\tilde{f}(\tilde{v})d\tilde{v} \\ &\geq -f(v)\left(F^{N-1}(p) + \int_p^v \tilde{f}(\tilde{v})d\tilde{v}\right) \\ &= -f(v)\left(F^{N-1}(v)\right) \\ &= \frac{dRHS}{dv}\end{aligned}$$

So both LHS and RHS decrease as v increases, but the RHS decreases faster at every v. As a result, it suffices to check whether $LHS(1) \leq RHS(1)$ to establish that $LHS(v) \leq RHS(v)$ for all $v \in [\hat{v}_t, 1]$. It is clear that $LHS(1) \leq RHS(1)$ holds since both are equal to 0. So, we conclude that the Border inequalities are satisfied with $\hat{v}_t = p$. □

Observe that the LHS and RHS of the Border inequalities are continuous in $\hat{v}_t$ for $\hat{v}_t \in (0,1)$, so let $\hat{v}_t$ be the infimal threshold v for which the Border inequalities are satisfied. The Border inequalities are satisfied at this $\hat{v}_t$, so the infimal $\hat{v}_t$ is the minimum $\hat{v}_t$ at which the inequalities hold.

Lemma 11. *If v_t is interior, then the Border constraint binds at v_t.*

Proof. Suppose the Border constraint did not bind at v_t so that

$$\frac{1-F(v_t)^N}{N} - \int_{v_t}^{1} \bar{q}(\tilde{v})dF(\tilde{v}) > 0$$

I will show that if $v_t > 0$, there exists $v_t' < v_t$ such that the Border constraints are satisfied.

$$\begin{aligned}\frac{d}{dv_t}\Big[\frac{1-F(v)^N}{N} - \int_{v}^{1} \bar{q}(\tilde{v})dF(\tilde{v})\Big] &= -\frac{d\bar{q}(v_t)}{dv_t}(1-F(v)) + \frac{p}{v_t}\tilde{f}(v_t)(1-F(v)) \\ &= (1-F(v))\frac{F^{N-1}(v_t)}{v_t^2}p \geq 0\end{aligned}$$

Further, we can express the difference between the RHS and LHS of the Border inequality for v as:

$$\frac{1-F(v)^N}{N} - \int_{v}^{1} \bar{q}(\tilde{v})dF(\tilde{v}) = \frac{1-F(v_t)^N}{N} - \int_{v_t}^{1} \bar{q}(\tilde{v})dF(\tilde{v}) + \int_{v_t}^{v} \frac{d}{d\hat{v}}\bigg(\frac{1-F(\hat{v})^N}{N} - \int_{\hat{v}}^{1} \bar{q}(\tilde{v})dF(\tilde{v})\bigg)d\hat{v}$$

Let's rewrite the last term:

$$\int_{v_t}^{v} \frac{d}{d\hat{v}}\bigg(\frac{1-F(\hat{v})^N}{N} - \int_{\hat{v}}^{1} \bar{q}(\tilde{v})dF(\tilde{v})\bigg)d\hat{v} = \int_{v_t}^{v} f(\hat{v})\bigg(\tilde{F}(v_t)(\frac{p}{v_t}-1) + \int_{v_t}^{\hat{v}}(\frac{p}{\tilde{v}}-1)\tilde{f}(\tilde{v})d\tilde{v}\bigg)d\hat{v}$$

Our goal is to show that $\exists v_t' < v_t$ such that

$$\frac{1-F(v)^N}{N} - \int_{v}^{1} \bar{q}(\tilde{v})dF(\tilde{v}) + \int_{v_t}^{v_t'} \frac{d}{dv_t}\Big[\frac{1-F(v)^N}{N} - \int_{v}^{1} \bar{q}(\tilde{v})dF(\tilde{v})\Big] \geq 0 \quad \forall v \geq v_t$$

We can rewrite the above inequality as:

$$\frac{1-F(v)^N}{N} - \int_{v}^{1} \bar{q}(\tilde{v})dF(\tilde{v}) \geq \int_{v_t'}^{v_t} \frac{d}{dv_t}\Big[\frac{1-F(v)^N}{N} - \int_{v}^{1} \bar{q}(\tilde{v})dF(\tilde{v})\Big] = (1-F(v))\int_{v_t'}^{v_t} \frac{F^{N-1}(\tilde{v}_t)}{\tilde{v}_t^2}pd\tilde{v}_t$$

The RHS is given by

$$\int_{v_t}^{v} f(\hat{v})\bigg(\tilde{F}(v_t)(\frac{p}{v_t}-1) + \int_{v_t}^{\hat{v}}(\frac{p}{\tilde{v}}-1)\tilde{f}(\tilde{v})d\tilde{v}\bigg)d\hat{v} + \frac{1-F(v_t)^N}{N} - \int_{v_t}^{1} \bar{q}(\tilde{v})dF(\tilde{v})$$

This function of $v \in [v_t, 1]$ is single-peaked since the derivative with respect to v is

$$f(v)(\tilde{F}(v_t)(\frac{p}{v_t}-1) + \int_{v_t}^{v}(\frac{p}{\tilde{v}}-1)\tilde{f}(\tilde{v})d\tilde{v}$$

This derivative is strictly positive at $v = v_t$ since $v_t \leq p$. Furthermore, this derivative is strictly increasing on (v_t, p) and strictly decreasing on $(p, 1)$. If the derivative ever crosses 0, it must do so on the interval $[p, 1]$ since the derivative starts strictly positive. Furthermore, this point is unique, if it exists. Finally, observe that such a point must exist because $\frac{1-F(1)^N}{N} - \int_1^1 \bar{q}(\tilde{v})dF(\tilde{v}) = 0$. If the derivative was ≥ 0 over $[v_t, 1]$, then we would have $\frac{1-F(1)^N}{N} - \int_1^1 \bar{q}(\tilde{v})dF(\tilde{v}) \geq \frac{1-F(v_t)^N}{N} - \int_{v_t}^1 \bar{q}(\tilde{v})dF(\tilde{v}) > 0$. So, call the unique point at which the derivative crosses 0, v^{th}. $v^{th} < 1$ by this argument.

Finally, we can choose $\varepsilon > 0$ small enough that $1 - \varepsilon > v^{th}$. The derivative is strictly negative over the interval $[1-\varepsilon, 1]$, so define

$$k := \max_{\hat{v} \in [1-\varepsilon, 1]} \tilde{F}(v_t)(\frac{p}{v_t} - 1) + \int_{v_t}^{\hat{v}} (\frac{p}{\tilde{v}} - 1)\tilde{f}(\tilde{v})d\tilde{v} < 0$$

k is well-defined because the derivative is continuous and $[1-\varepsilon, 1]$ is compact. Consider the affine function through $(1, 0)$ with slope k. Say this affine function has value y at $1 - \varepsilon$. Now, define $y' := \min\{y, \frac{1-F(v_t)^N}{N} - \int_{v_t}^1 \bar{q}(\tilde{v})dF(\tilde{v})\} > 0$. Consider the affine function through $(1, 0)$ and (v_t, y'). Denote its slope by $-s$. Note that $s < 0$. This affine function is below $\frac{1-F(v)^N}{N} - \int_v^1 \bar{q}(\tilde{v})dF(\tilde{v})$ on the interval $[v_t, v^{th}]$ since $y' < \frac{1-F(v_t)^N}{N} - \int_{v_t}^1 \bar{q}(\tilde{v})dF(\tilde{v})$ and $\frac{1-F(v)^N}{N} - \int_v^1 \bar{q}(\tilde{v})dF(\tilde{v})$ is increasing on $[v_t, v^{th}]$. Next, for all $v \in [v^{th}, 1-\varepsilon]$, $\frac{1-F(v)^N}{N} - \int_v^1 \bar{q}(\tilde{v})dF(\tilde{v})$ is decreasing so $\frac{1-F(v)^N}{N} - \int_v^1 \bar{q}(\tilde{v})dF(\tilde{v}) \geq y \geq y'$ and $\frac{1-F(v)^N}{N} - \int_v^1 \bar{q}(\tilde{v})dF(\tilde{v})$ is above the affine function through $(1, 0)$ and (v_t, y'). Finally, the affine function through $(1, 0)$ and (v_t, y') is weakly below the affine function through $(1, 0)$ and $(1-\varepsilon, y)$. The affine function through $(1, 0)$ and $(1-\varepsilon, y)$ is by construction below $\frac{1-F(v)^N}{N} - \int_v^1 \bar{q}(\tilde{v})dF(\tilde{v})$ on $[1-\varepsilon, 1]$.

Next, choose $\delta > 0$. Consider $\frac{1-F(v)^N}{N} - \int_v^1 \bar{q}(\tilde{v})dF(\tilde{v})$ for $v \in [v_t - \delta, v_t]$. On this interval, we have that

$$\begin{aligned}\frac{1-F(v)^N}{N} - \int_v^1 \bar{q}(\tilde{v})dF(\tilde{v}) &= \frac{1-F(v)^N}{N} - \int_{v_t}^1 \bar{q}(\tilde{v})dF(\tilde{v}) \\ &> \frac{1-F(v_t)^N}{N} - \int_{v_t}^1 \bar{q}(\tilde{v})dF(\tilde{v}) > 0\end{aligned}$$

We would like to show that $\exists v_t'' < v_t$ such that

$$\begin{aligned}\frac{1-F(v)^N}{N} - \int_v^1 \bar{q}(\tilde{v})dF(\tilde{v}) \geq \int_{v_t''}^{v_t} &\mathbb{I}\{\tilde{v}_t \leq v\}\left[\frac{p}{\tilde{v}_t^2}\tilde{f}(\tilde{v}_t)(1-F(v))\right] \\ &+ \mathbb{I}\{\tilde{v}_t > v\}\left[\frac{p}{\tilde{v}_t^2}\tilde{f}(\tilde{v}_t)(1-F(\tilde{v}_t)) + \tilde{F}(\tilde{v}_t)f(\tilde{v}_t)\right]d\tilde{v}_t\end{aligned}$$

To this end, define for all $\tilde{v}_t \in [v_t - \delta, v_t]$

$$\kappa(\tilde{v}_t) := \sup_{v\in[v_t-\delta,v_t]} \mathbb{I}\{\tilde{v}_t \leq v\}\left[\frac{p}{\tilde{v}_t^2}\tilde{f}(\tilde{v}_t)(1-F(v))\right] + \mathbb{I}\{\tilde{v}_t > v\}\left[\frac{p}{\tilde{v}_t^2}\tilde{f}(\tilde{v}_t)(1-F(\tilde{v}_t)) + \tilde{F}(\tilde{v}_t)f(\tilde{v}_t)\right]$$

Notice that $\kappa(\tilde{v}_t) > 0$ since f and $\tilde{f}$ are continuous and so have maximum values bounded away from 0 on this interval.

$$\int_{v_t''}^{v_t} \kappa(\tilde{v}_t)d\tilde{v}_t \geq \int_{v_t''}^{v_t} \mathbb{I}\{\tilde{v}_t \leq v\}\left[\frac{p}{\tilde{v}_t^2}\tilde{f}(\tilde{v}_t)(1-F(v))\right] + \mathbb{I}\{\tilde{v}_t > v\}\left[\frac{p}{\tilde{v}_t^2}\tilde{f}(\tilde{v}_t)(1-F(\tilde{v}_t)) + \tilde{F}(\tilde{v}_t)f(\tilde{v}_t)\right]d\tilde{v}_t$$

for each $v_t'' \in [v_t - \delta, v_t]$.

$$\begin{aligned}\frac{1-F(v)^N}{N} - \int_v^1 \bar{q}(\tilde{v})dF(\tilde{v}) > \frac{1-F(v_t)^N}{N} &- \int_{v_t}^1 \bar{q}(\tilde{v})dF(\tilde{v}) \\ &\geq \int_{v_t''}^{v_t} \kappa(\tilde{v}_t)d\tilde{v}_t\end{aligned}$$

Since the RHS of this inequality is continuous in v_t'', we can pick a v_t'' sufficiently small to make the above inequality hold.

Define $s' = \min\{s, \max_{v\in[v_t-\delta,v_t]} F(v)^{N-1}f(v)\} > 0$. Finally, observe that we can choose $v_t' < v_t$ so that $v_t - v_t' < \delta$, $v_t' < v_t''$ and

$$\frac{1-F(v_t)^N}{N} - \int_{v_t}^1 \bar{q}(\tilde{v})dF(\tilde{v}) \geq (1-F(v_t))\int_{v_t'}^{v_t} \frac{F^{N-1}(\tilde{v}_t)}{\tilde{v}_t^2}pd\tilde{v}_t$$

and

$$f(v_t)\int_{v_t'}^{v_t} \frac{F^{N-1}(\tilde{v}_t)}{\tilde{v}_t^2}pd\tilde{v}_t \leq s'$$

This is sufficient for

$$\frac{1-F(v)^N}{N} - \int_v^1 \bar{q}(\tilde{v})dF(\tilde{v}) \geq (1-F(v))\int_{v_t'}^{v_t} \frac{F^{N-1}(\tilde{v}_t)}{\tilde{v}_t^2}pd\tilde{v}_t \quad \forall v \in [v_t', 1]$$

We have shown the claim. □

C.2 Proposition 5

First, observe that if the principal sets a mechanism that induces a linear cost function with slope $p \leq 0$, then in the worst case collusive outcome, the principal makes weakly negative revenue. For any collusive arrangement $(\rho, \tilde{q}, \tilde{t})$, colluders can lower the principal's revenue by coordinating on aggregate outcomes with expected cost arbitrarily close to the cost function. The cost function is weakly negative for all $Q \in [0,1]$, so the worst-case revenue for the principal is weakly negative.

Next, observe that if the principal's mechanism induces a cost function with slope $p \geq 1$, then the joint surplus of the truth-telling BNE must be 0. The ex post joint surplus of bidders playing the BNE is at most 0. If the bidder with the highest valuation receives the good with probability Q, this generates a surplus of $Q \max_i v_i$, but whenever the good is acquired with probability Q, bidders pay in total an amount that is weakly greater than $Q \max_i v_i$. So, the joint ex post surplus of bidders playing the BNE is weakly less than 0 for all value profiles v. As a result, all bidders playing the opt-out action is a feasible collusive arrangement and results in a revenue of 0 for the principal.

Thus, for the principal to make positive worst-case revenue, he must set a mechanism with $p \in (0,1)$; as long as such a mechanism results in a positive BNE surplus, then the principal will make strictly positive worst-case revenue when facing colluders since colluder must purchase from the principal in order to generate positive surplus from collusion and beat the surplus from non-collusive play. The rest of the argument proceeds by solving the f-Surplus Max problem for $f(Q) = pQ$ while additionally requiring that $t \geq 0$. The condition of Proposition 5 implies that each $p \in (0,1)$ results in an interior v_t; for each $p \in (0,1)$, it is possible to verify that the mechanism given in the statement of the proposition solves the f-Surplus Max problem. I do that by using weak duality.

Let's write the Lagrangian:

$$\begin{aligned}\mathcal{L}(q,t,\lambda,\gamma,\mu,\beta) = &\int_v q_i(v)v_i - t_i(v)dF(v) + \sum_i \int_{v_i} \lambda_i(v_i)\Big[-\frac{d\bar{q}_i}{d\tilde{v}_i}(\tilde{v}_i)\tilde{v}_i + \frac{d\bar{t}_i}{d\tilde{v}_i}\Big]dv_i \\ &+ \sum_i \int_{\tilde{v}_i} \gamma_i \Big[\bar{q}_i(\tilde{v}_i)\tilde{v}_i - \bar{t}_i(\tilde{v}_i)\Big] d\tilde{v}_i \\ &+ \int_v \mu(v)(\sum_i t_i(v) - p\sum_i q_i(v))dv + \int_v \beta(v)(1 - \sum_i q_i(v))dv\end{aligned}$$

To construct the multipliers, first define some objects:

$$C := \frac{p\int_{v_t}^1 \hat{v} dF_i(\hat{v})}{f_i(v_t)(v_t)(p - v_t) + p(1 - F(v_i))}$$

Choose the multipliers on the local IC to be

$$\lambda(v_i) = \frac{\int_{v_i}^{1} \hat{v} - C dF_i(\hat{v})}{v_i}$$

for the subset $v_i \in [v_t^{IR}, 1]$ with v_t^{IR} given by the solution to:

$$\frac{\int_{v_t^{IR}}^{1} \hat{v} dF_i(\hat{v})}{1 - F_i(v_t^{IR})} = C$$

Observe that $v_t^{IR} \leq v_t$ since

$$C = \frac{p \int_{v_t}^{1} \hat{v} dF_i(\hat{v})}{f_i(v_t)(v_t)(p - v_t) + p(1 - F(v_i))} \leq \frac{\int_{\hat{v}_t}^{1} \hat{v} dF_i(\hat{v})}{(1 - F(\hat{v}_t))}$$

since $p \geq \hat{v}_t$ and additionally $\frac{\int_x^1 \hat{v} dF_i(\hat{v})}{1-F(x)}$ is increasing in x. Notice also that $\lambda(v_i)$ is clearly single-peaked at $v_i = C$. Another feature of v_t^{IR} is that $\lambda(v_t^{IR}) = 0$.

With these objects, we study a particular further relaxation, setting the local IC multipliers λ to be 0 for $v_i < v_t^{IR}$ and setting the IR multipliers γ to be 0 for $v_i \geq v_t^{IR}$:

$$\begin{aligned}
\mathcal{L}(q, t, \lambda, \gamma, \mu, \beta) = & \int_v q_i(v) v_i - t_i(v) dF(v) + \sum_i \int_{v_t^{IR}}^{1} \lambda_i(v_i) \Big[- \frac{d\bar{q}_i}{d\tilde{v}_i}(\tilde{v}_i)\tilde{v}_i + \frac{d\bar{t}_i}{d\tilde{v}_i} \Big] dv_i \\
& + \sum_i \int_0^{v_t^{IR}} \gamma_i(\tilde{v}_i) \Big[\bar{q}_i(\tilde{v}_i)\tilde{v}_i - \bar{t}_i(\tilde{v}_i) \Big] d\tilde{v}_i \\
& + \int_v \mu(v) (\sum_i t_i(v) - p \sum_i q_i(v)) dv + \int_v \beta(v) (1 - \sum_i q_i(v)) dv
\end{aligned}$$

Integration by parts and then collecting the coefficients of all $q_i(v)$ and $t_i(v)$ variables gives:

$$\begin{aligned}
-f(v) - f_{-i}(v_{-i})\lambda_i'(v_i) + \mu(v) \leq 0 \quad & \text{if } v_i \geq v_t^{IR} \\
-f(v) - f_{-i}(v_{-i})\gamma_i(v_i) + \mu(v) \leq 0 \quad & \text{if } v_i < v_t^{IR} \\
f(v)(v_i) + f_{-i}(v_{-i})(\lambda_i(v_i) + \lambda_i'(v_i)v_i) - p\mu(v) - \beta(v)\mathbb{I}\{\max v \geq v_t\} \leq 0 \quad & \text{if } v_i \geq v_t^{IR} \\
f(v)(v_i) + f_{-i}(v_{-i})\gamma_i(v_i)v_i - p\mu(v) - \beta(v)\mathbb{I}\{\max v \geq v_t\} \leq 0 \quad & \text{if } v_i < v_t^{IR}
\end{aligned}$$

We seek to find dual multipliers such that the mechanism in the proposition statement is optimal. So, if $\max_j v_j \geq v_t$, we would like the $q_i(v)$ constraint to bind for any i where $v_i \geq v_t$. Given our choice of local IC multipliers, we have that that $f(v)(v_i) + f_{-i}(v_{-i})(\lambda_i(v_i) +$

$\lambda_i'(v_i)v_i) = f(v)C$ for all v as long as $v_i \geq v_t^{IR}$. As a result, we can choose $\beta(v)$ to be

$$\beta(v) = f(v)C - p\mu(v)$$

for all v s.t. $\max_j v_j \geq v_t$ and $\mu(v)$ is chosen for all v such that $\max_j v_j \geq v_t^{IR}$ so that $t_i(\max_j v_j)$ binds where i is the bidder with the highest realized valuation (which is above v_t by assumption):

$$\mu(v) = f(v)(1 + \lambda_i'(v_i)/f_i(v_i))$$

Otherwise, $\mu(v) = f(v)(1 + \lambda_i'(v_t^{IR})/f_i(v_t^{IR}))$. Furthermore, $\gamma_i(v_i)$ is chosen so that

$$\gamma_i(v_i) = f_i(v_i)\lambda_i'(v_t^{IR})/f_i(v_t^{IR})$$

This is a full specification of candidate dual multipliers. To verify dual feasibility, we go constraint by constraint:

1. $-f(v) - f_{-i}(v_{-i})\lambda_i'(v_i) + \mu(v) \leq 0$ if $v_i \geq v_t^{IR}$. This will be verified if we have that

$$1 + \lambda_i'(\max_j v_j)/f_i(\max_j v_j) \leq 1 + \lambda_i'(v_i)/f_i(v_i)$$

2. $-f(v) - f_{-i}(v_{-i})\gamma_i(v_i) + \mu(v) \leq 0$ if $v_i < v_t^{IR}$. This is equivalent to

$$\begin{aligned} 1 + \lambda_i'(v_i^{IR})/f_i(v_t^{IR}) &\leq 1 + \lambda_i'(v_t^{IR})/f_i(v_t^{IR}) \quad \text{if } \max_j v_j < v_t^{IR} \\ 1 + \lambda_j'(v_j)/f_j(v_j) &\leq 1 + \lambda_i'(v_i^{IR})/f_i(v_t^{IR}) \quad \text{if } \max_j v_j > v_t^{IR} \end{aligned}$$

   The second inequality will hold if $1 + \lambda_i'(v_i)/f_i(v_i)$ is decreasing in v_i.

3. $f(v)(v_i) + f_{-i}(v_{-i})(\lambda_i(v_i) + \lambda_i'(v_i)v_i) - p\mu(v) - \beta(v)\mathbb{I}\{\max v \geq v_t\} \leq 0$ if $v_i \geq v_t^{IR}$. Now, if $\max v \geq v_t$ and $v_t \geq v_t^{IR}$, this inequality holds because $f(v)(v_i) + f_{-i}(v_{-i})(\lambda_i(v_i) + \lambda_i'(v_i)v_i)$ was constructed to be $f(v)C$. If $\max v < v_t$, then $\mu(v) = f(v)(1+\lambda_i'(v_t^{IR})/f_i(v_t^{IR}))$, we should have

$$\begin{aligned} f(v)C - pf(v)(1 + \lambda_i'(v_t^{IR})/f_i(v_t^{IR})) &\leq 0 \\ C &\leq p(1 + \lambda_i'(v_t^{IR})/f_i(v_t^{IR})) \end{aligned}$$

which holds because

$$
\begin{aligned}
1 + \lambda_i'(v_t^{IR})/f_i(v_t^{IR}) &= \frac{C}{v_t^{IR}} - \frac{\int_{v_t^{IR}}^1 \hat{v} - C dF_i(\hat{v})}{f_i(v_t^{IR})(v_t^{IR})^2} \\
&= \frac{C}{v_t^{IR}} - \frac{(1 - F_i(v_t^{IR}))(C - C)}{f_i(v_t^{IR})(v_t^{IR})^2} \\
&= \frac{C}{v_t^{IR}}
\end{aligned}
$$

and $v_t^{IR} \leq v_t \leq p$.

4. $f(v)(v_i) + f_{-i}(v_{-i})\gamma_i(v_i)v_i - p\mu(v) - \beta(v)\mathbb{I}\{\max v \geq v_t\} \leq 0$ if $v_i < v_t^{IR}$. If we have that $\max v \geq v_t$, we have that

$$
\begin{aligned}
f(v)v_i + f(v)\lambda_i'(v_t^{IR})/f_i(v_t^{IR})v_i - p\mu(v) &\leq f(v)C - p\mu(v) \\
v_i + \lambda_i'(v_t^{IR})/f_i(v_t^{IR})v_i &\leq C = v_t^{IR}(1 + \lambda'(v_t^{IR})/f_i(v_t^{IR}))
\end{aligned}
$$

which follows since $v_i \leq v_t^{IR}$. Now, suppose $\max v < v_t$. We would like to show that

$$
\begin{aligned}
f(v)v_i + f(v)\lambda_i'(v_t^{IR})/f_i(v_t^{IR})v_i - p\mu(v) &\leq 0 \\
v_i + \lambda_i'(v_t^{IR})/f_i(v_t^{IR})v_i &\leq p + \lambda_i'(v_t^{IR})/f_i(v_t^{IR})p
\end{aligned}
$$

which holds since $v_i \leq v_t^{IR} \leq p$.

Let's go variable by variable to verify that multipliers are weakly positive:

1. Local IC multipliers:

$$
\lambda(v_i) \geq 0
$$

is easy to verify because $\lambda(\cdot)$ is single-peaked and v_t^{IR} is set at the value where $\lambda(\cdot)$ first becomes positive.

2. IR multipliers:

$$
\gamma_i(v_i) = f_i(v_i)\lambda_i'(v_t^{IR})/f_i(v_t^{IR})
$$

again comes from the single-peakedness of $\lambda(\cdot)$. At v_t^{IR}, λ_i is increasing.

3. Menu cost: If

$$1 + \lambda_i'(\tilde{v})/f_i(\tilde{v})$$

is decreasing in $\tilde{v}$, then it is sufficient to check that $1 + \lambda_i'(1)/f_i(1) \geq 0$ to get that $\mu(v) \geq 0$ for all v. In the end, $1 + \lambda_i'(1)/f_i(1) = C$.

4. Feasibility:

$$f(v)C - p\mu(v) \leq 0$$
$$\frac{C}{p} \leq 1 + \lambda_i'(\max v)/f_i(\max v)$$

Again, if $1 + \lambda_i'(\tilde{v})/f_i(\tilde{v})$ is decreasing in $\tilde{v}$, we just need to check that $C/p \leq 1 + \lambda_i'(\max v)/f_i(\max v)$ holds for $\max v = v_t^{IR}$ which we checked in point 2 of the dual feasibility check.

To complete the verification that the dual variables are weakly positive and satisfy dual feasibility, observe that $1 + \lambda_i'(v_i)/f_i(v_i)$ is decreasing in v_i since

$$\begin{aligned} 1 + \lambda_i'(\tilde{v})/f_i(\tilde{v}) &= \frac{C}{\tilde{v}} - \frac{\int_{\tilde{v}}^1 \hat{v} - C dF_i(\hat{v})}{f_i(\tilde{v})(\tilde{v})^2} \\ &= \frac{C}{\tilde{v}} + \frac{1}{\tilde{v}^2}\frac{1 - F_i(\tilde{v})}{f_i(\tilde{v})}\big(C - \mathbb{E}[\hat{v}|\hat{v} \geq \tilde{v}]\big) \end{aligned}$$

Notice that the claim follows since $C/\tilde{v}$, $1/\tilde{v}^2$, $(1 - F_i(\tilde{v}))/f_i(\tilde{v})$, and $C - \mathbb{E}[\hat{v}|\hat{v} \geq \tilde{v}]$ are all decreasing in $\tilde{v}$.

The mechanism given in the proposition statement maximizes the Lagrangian fixing these dual variables since it sets $q_i(v)$ to be 0 whenever $v_i < v_t$ and $t_i(v)$ to be 0 whenever v_i is not the maximum valuation or the maximum valuation is weakly lower than v_t.

The last step to showing that the mechanism is optimal is to verify complementary slackness. Notice that the local IC constraints hold at equality due to the payoff formula. Furthermore, the IR constraints hold at equality for all $v_i < v_t^{IR}$. Next, observe that if we could show that $1 = \sum_i q_i(v)$ for almost all v where $\max v \geq v_t$, then we have finished verifying complementary slackness since for almost all v, we would have $\sum_i t_i(v) = p \sum_i q_i(v)$ and $1 = \sum_i q_i(v)$. Since v_t is interior for any $p \in (0, 1)$, then Lemma 11 tells us that the Border inequality binds at v_t. This implies that

$$Pr(i \text{ gets the good}) = \int_{v_t}^1 \bar{q}(\tilde{v})dF(\tilde{v}) = \frac{1 - F(v_t)^N}{N} = \frac{1}{N} Pr(\max v \geq v_t)$$

Since the interim allocation is symmetric, we have that this implies

$$Pr(\text{any bidder gets the good}) = Pr(\max_j v_j \geq v_t)$$

The interim allocation function implies that the good is not allocated to any bidder when $\max v < v_t$, so verifying that the Border constraint binds at v_t is enough to show that $1 = \sum_i q_i(v)$ holds with probability 1 given that $\max_j v_j \geq v_t$.

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